```
    tmx=[];
    tmy=[];
    for j=1:n,
      tmx=[tmx;v{i}{1,j}(1,1)]; tmy=[tmy;v{i}{1,j}(1,2)];
    end
    tmx=[tmx;v{i}{1,1}(1,1)]; tmy=[tmy;v{i}{1,1}(1,2)]; plot(tmx,tmy);
  end
  plot([0,sz,sz,0,0],[0,0,sz,sz,0]); axis equal; axis off; wen=~cmpc(n,sz,x,rad,v);
end
N0=num-stp; N1=num; tmn=0;
for l=1:k,
  cnt=cnt+1; num=(N0+N1)/2; tmx=x(1:num,:); tmr=rad(1:num,:); pced=cmpc(n,sz,tmx,tmr,v);
  if(pced)
    N1=num; tmn=0;
  else
    N0=num; tmn=tmn+1;
  end
  figure(cnt);clf; hold on;
  for i=1:num,
    tmx=[]; tmy=[];
    for j=1:n,
      tmx=[tmx;v{i}{1,j}(1,1)]; tmy=[tmy;v{i}{1,j}(1,2)];
    end
    tmx=[tmx;v{i}{1,1}(1,1)]; tmy=[tmy;v{i}{1,1}(1,2)]; plot(tmx,tmy);
  end
  plot([0,sz,sz,0,0],[0,0,sz,sz,0]); axis equal; axis off;
end
% ppgt.m, threshold area ratio, Kit Tiyapan, (c) 20th November, 2002
clear all; rand(’state’,sum(100*clock)); n=5; sz=10; stp=16; wen=1;  % wiederholen
num=0; cnt=0; x=[]; rad=[];
while wen
  cnt=cnt+1; num=num+stp; x=[x;sz*rand(16,2)]; ang=2*pi/n; tpi=2*pi;
  rad=[rad;tpi*rand(stp,1)]; r=sqrt(1/(n*sin(ang/2)*cos(ang/2)));
  for i=(num-stp+1):num,
    for j=0:(n-1),
      tmp=rad(i,1)+j*ang;
      v{i}{1,(j+1)}=[(x(i,1)+r*cos(tmp)),(x(i,2)+r*sin(tmp))];
    end
  end
  figure(cnt);clf; hold on;
  for i=1:num,
    tmx=[]; tmy=[];
    for j=1:n,
      tmx=[tmx;v{i}{1,j}(1,1)]; tmy=[tmy;v{i}{1,j}(1,2)];
    end
    tmx=[tmx;v{i}{1,1}(1,1)]; tmy=[tmy;v{i}{1,1}(1,2)]; plot(tmx,tmy);
  end
  plot([0,sz,sz,0,0],[0,0,sz,sz,0]); axis equal; axis off; wen=~cmpc(n,sz,x,rad,v);
end
N0=num-stp; N1=num;
for k=1:4,
  cnt=cnt+1; num=(N0+N1)/2; tmx=x(1:num,:); tmr=rad(1:num,:);
  pced=cmpc(n,sz,tmx,tmr,v);
  if(pced)
    N1=num;
  else
    N0=num;
  end
  figure(cnt);clf; hold on;
  for i=1:num,
    tmx=[]; tmy=[];
    for j=1:n,
      tmx=[tmx;v{i}{1,j}(1,1)]; tmy=[tmy;v{i}{1,j}(1,2)];
    end
    tmx=[tmx;v{i}{1,1}(1,1)]; tmy=[tmy;v{i}{1,1}(1,2)]; plot(tmx,tmy);
  end
  plot([0,sz,sz,0,0],[0,0,sz,sz,0]); axis equal; axis off;
end
% cmpc.m, a function, Kit Tiyapan, (c) 20th November, 2002
function [percolated] = cmpc(n,Size,X,Rad,V);
% 2-d continuum percolation of n-gons
N=size(X,1); Angle=2*pi/n; TwoPi=2*pi; R=sqrt(1/(n*sin(Angle/2)*cos(Angle/2)));
Tmp=V{1}{1,1}+(V{1}{1,2}-V{1}{1,1})/2; dx=Tmp(1,1)-X(1,1); dy=Tmp(1,2)-X(1,2);
r=sqrt(dx*dx + dy*dy); T=delaunay(X(1:N,1),X(1:N,2)); NT=size(T,1);
D=sparse(N,N); Ov=sparse(N,N); Ov1=sparse(N,1); Pair=[]; Limbo=[];
Oclock=sparse(NT,NT);
for i=1:NT,
  Tmp=[T(i,:),T(i,1)];
  for j=1:3,
    c1=Tmp(1,j); c2=Tmp(1,(j+1)); dx=X(c2,1)-X(c1,1); dy=X(c2,2)-X(c1,2);
    TmpA=sqrt(dx*dx + dy*dy); D(c1,c2)=TmpA; D(c2,c1)=TmpA; Pair=[Pair;[c1,c2;c2,c1]];
    if(TmpA<=(2*r))
```

```
      Ov(c1,c2)=1; Ov(c2,c1)=1; Ov1(c1,1)=1; Ov1(c2,1)=1;
    elseif(TmpA<=(2*R))
      Limbo=[Limbo;[c1,c2;c2,c1]];
    end
  end
  TmpB=atan(abs(dy/dx));
  if(dx>=0)
    if(dy>=0)  % Quadrant 1
      Oclock(c1,c2)=TmpB; Oclock(c2,c1)=pi+TmpB;
    else  % Quadrant 4
      Oclock(c1,c2)=TwoPi-TmpB; Oclock(c2,c1)=pi-TmpB;
    end
  else
    if(dy>=0)  % Quadrant 2
      Oclock(c1,c2)=pi-TmpB; Oclock(c2,c1)=TwoPi-TmpB;
    else  % Quadrant 3
      Oclock(c1,c2)=pi+TmpB; Oclock(c2,c1)=TmpB;
    end
  end
end
Tmp=Angle/2; Star=[];
for i=1:N,
  TmpA=[]; TmpB=Rad(i,1);
  for j=1:n,
    TmpA=[TmpA,mod((TmpB + (j-1)*Angle + Tmp),TwoPi)];
  end
  Star=[Star;TmpA];
end
Wobble=sparse(N,N); jWobble=sparse(N,N); TmpN=size(Limbo,1);
for i=1:TmpN,
  Min=10; TmpA=Limbo(i,1); TmpB=Limbo(i,2); jMin=j;
  for j=1:n,
    Tmp=Star(TmpA,j)-Oclock(TmpA,TmpB);
    if(abs(Tmp)<abs(Min))
      Min=Tmp; jMin=j;
    end
  end
  Wobble(TmpA,TmpB)=Min; jWobble(TmpA,TmpB)=jMin;
end
Tmp=Angle/2;
for i=1:2:TmpN,
  TmpA=Limbo(i,1); TmpB=Limbo(i,2);
  if(abs(Wobble(TmpB,TmpA)) >= abs(Wobble(TmpA,TmpB)))
    TmpA=Limbo((i+1),1); TmpB=Limbo((i+1),2);
  end
  J=jWobble(TmpA,TmpB); v{1}=V{TmpA}{1,J};
  if(J==n)
    v{2}=V{TmpA}{1,1};
  else
    v{2}=V{TmpA}{1,(J+1)};
  end
  J=jWobble(TmpB,TmpA); v{3}=V{TmpB}{1,J};
  if(J==n)
    v{4}=V{TmpB}{1,1};
  else
    v{4}=V{TmpB}{1,(J+1)};
  end
  Max=0;
  if(Wobble(TmpA,TmpB)>=0)
    vMin=v{1};
  else
    vMin=v{2};
  end
  d1=R*cos(Tmp-abs(Wobble(TmpA,TmpB))); TmpD=(X(TmpA,1)-X(TmpB,1));
  a=(X(TmpA,2)-X(TmpB,2))/TmpD; b=(X(TmpA,1)*X(TmpB,2)-X(TmpB,1)*X(TmpA,2))/TmpD;
  a1=a; b1=vMin(1,2)-a1*vMin(1,1); x3=v{3}(1,1); y3=v{3}(1,2); x4=v{4}(1,1);
  y4=v{4}(1,2); TmpD=x3-x4; p=(y3-y4)/TmpD; q=(x3*y4-x4*y3)/TmpD;
  TmpD=a1-p;
  x=(q-b1)/TmpD; y=(a1*q-b1*p)/TmpD; dx=x-X(TmpB,1); dy=y-X(TmpB,2);
  r2=sqrt(dx*dx + dy*dy); d2=r2*cos(Tmp-abs(Wobble(TmpB,TmpA)));
  d=D(TmpA,TmpB);
  if((d1+d2)>=d)
    Ov(TmpA,TmpB)=1; Ov(TmpB,TmpA)=1; Ov1(TmpA,1)=1; Ov1(TmpB,1)=1;
  end
end
Clus=Ov;
for i=1:N,
  Clus(i,i)=1;
end
NClus=size(Clus,1); ClusA=Clus(1,:); NClusA=1;
for i=2:NClus,
  Joined=0;
  for j=1:NClusA,
    TmpC=Clus(i,:) | ClusA(j,:);
    if(sum(Clus(i,:) & ClusA(j,:)))
      ClusA(j,:)=TmpC; ClusB=ClusA; ClusA=TmpC; NClusB=NClusA;
      NClusA=1; Joined=1; break;
    end
  end
  if(~Joined)
    ClusA=[ClusA;Clus(i,:)]; NClusA=NClusA+1;
  else
    for j=1:NClusB,
      if(sum(ClusA(1,:) & ClusB(j,:)))
        ClusA(1,:)=ClusA(1,:) | ClusB(j,:);
      else
        ClusA=[ClusA;ClusB(j,:)]; NClusA=NClusA+1;
      end
    end
  end
end
Left=sparse(1,N); Right=sparse(1,N); Margin=0.1*Size;
for i=1:N,
  if(X(i,1)<=Margin)
    Left(1,i)=1;
  elseif(X(i,1)>=(Size-Margin))
    Right(1,i)=1;
  end
end
percolated=0;
for i=1:NClusA,
  if(sum(Left & ClusA(i,:)) & sum(Right & ClusA(i,:)))
    percolated=1; break;
  end
end
```

B

§ B. Terminology and other resources

a, A

aboulia. lack of will or initiative seen with organic disease or damage to the brain.

adjacent. have a certain thing in common. ~ **edges**, edges which which have a common vertex. ~ **tiles**, tiles which have a common edge. ~ **vertices**, vertices which have a common edge.

affine. (*Lat.* affinis; *Fr.* affin) relating to a coordinate transformation that is equivalent to a linear transformation followed by a translation. **affine combination**, $p = \sum_1^k \alpha_i p_i$, where $p_1, p_2, \ldots, p_k$ are points in E^d, $\alpha_i = \mathrm{Re}$ and $\sum_{i=1}^k \alpha_i = 1$. ~ **geometry**, studies the properties which are preserved (invariant) under transformations in the affine group where A in $x' = xA + c$ is nonsingular. ~ **hull**, the smallest affine set containing L, where L is a subset of E^d. ~**ly independent points**, $p_i \in E^d, \alpha_i \in \Re, i = 1, 2, \ldots, k$ where $p_2 - p_1, \ldots, p_k - p_1$ linearly independent. ~ **mapping**, is $x' = xA + c$. ~ **set**, a linear combination $p = \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k$ where $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$, an affine set is the translation of a linear set vector subspace or it is simply *flat.*

akinesia. total lack of movement.

alternate. ~ **interior angles**, those angles which lie on the opposite sides of a transversal.

altitude. the line from the vertex of a triangle perpendicular to its opposite side.

anorexia. violent refusal to eat.

antipodal points. those points which admit no parallel supporting lines.

Archimedean polyhedron. a polyhedron whose faces are all regular polygons and whose vertices are all congruent to one another.

arteriosclerosis. A general term for the thickening, hardening, and loss of elasticity of the walls of blood vessels.

atherosclerosis. from *Gr. athero* (gruel, paste) and *sclerosis* (hardness). An arteriosclerosis which is caused by the deposition of materials, for example calcium, cellular waste products, cholesterol, fatty substances and fibrin, on the inner lining of an artery. If occurs at a carotid artery it can cause a stroke, while if at a coronary artery a heart attack.

augmented. (of polyhedra) having one or more k-sided faces replaced by a k-gonal pyramid, cupolar, or rotunda.

automatism. forced obedience to external command.

b, B

bimedian. the line joining the mid points of two opposite sides of a quadrilateral.

block. resistance to movement or thought at any level.

bond. link between two cells which share a face.

boundary. ~ **of a ball**, $\partial N_\epsilon(c) = \{x | |x - c| = \epsilon\}$ for $c \in R^m, \epsilon > 0$, a hypersphere in R^m.

bulimia. a violent and insatiable appetite .

c, C

cataclasis. granulation.

cataclasite. rock deformed by shearing and cataclasis.

categorical system. an axiomatic sytem S where each pair of its models is isomorphic with respect to S.

central projection. the one-to-one correspondence between points of the plane $x_{d+1} = 1$ (ie. a space E^d) and points on the hemisphere of S^{d+1} corresponding to $x_{d+1} > 0$.

chain. a planar straight line graph $C = (u_1, u_2, \ldots, u_p)$ with vertex set $\{u_1, u_2, \ldots, u_p\}$ and edge set (u_i, u_{i+1}), $i = 1, 2, \ldots, p - 1$.

-cingulum. *sfx* a belt of 12 triangles.

circumcentre. the centre of a circumscribed circle. ~ **of a triangle**, the point of concurrency of the perpendicular bisectors of the three sides of it.

circumcircle. or circumscribed circle is the circle which contains the three vertices of a triangle and has the circumcentre of that triangle as its centre.

claim. a small theorem to be proved, often it is a theorem presented and proved within the proof of another theorem.

close. ~**d ball**, $\mathcal{N}_\epsilon(c) = \{x | |x - c| < \epsilon\}$ for $c \in R^m, \epsilon > 0$. **vertex of** ~ **type**,v where $|T \bigoplus T_1 \bigoplus T_2| = k + 2$ if $k = N - 1$, where vertex v is common to three polygons $V(T), V(T_1), V(T_2) \in \mathrm{Vn}_k(S)$.

combinatorial geometry. geometry which characterises the geometrical objects as properties of finite subsets.

complement. $A \backslash B = \{x | x \in A, x \notin B\}$.

completeness. (of an axiomatic system S) impossible to add an independent axiom.

component. connected subgraph.

concurrent. (*Lat.* concurrere) ~ **lines**, three or more line which intersect at the same point (point of concurrency).

congruence transformation. mapping of the Euclidean plane onto itself which preserves all distances.

congruent. (*Lat.* congruentem, *nom* congruens)~ **tilings**,tiles which coincide with each other via a rigid motion of the plane or reflection. ~ **triangle**,a triangle in which a one-to-one correspondence can be established between their vertices such that corresonding sides are congruent and corresponding angles are congruent. Two relevant theorems to the congruence of triangles are the SSS, SAS and ASA theorems.

connected. comprising of only one piece. ~ **graph**, G such that $\forall g, v_i, v_j \in G, \exists v_i g v_j \in G$.

consistency. (of an axiomatic system) making no claims of contradictory statements.

convex. (*Lat.* convexus) ~ **polygon**, the region on the true side of all the half-planes of its sides. Substitution of any point coordinates into all of its half-plane equations yields negative when that point is inside the polygon. ~ **hull**, the smallest convex set containing $P \in E^d$. It defines A and b such that $\forall x \in P, Ax+b \leq 0$. ~ **set**, is a set where a line segment formed by any pair of its points lies within the set.

coplanar points. points which lie on the same plane with each other.

corner. a vertex of a polygon, in order to distinguish itself from that of a tiling.

-corona. *sfx* a crownlike structure of eight triangles. **-mega**~, such a complex of 12 triangles.

corresponding angles. two angles, one exterior the other interior, which lie on the same side of the transversal.

covering. a family of sets which completely covers a plane. ~ **lattice**,a lattice derived from another lattice by exchanging edges with vertices, the position of each edge normally being taken to be that of its mid point.

Coxeter-Todd lattice. the lattice in 12 dimensions which has the maximum packing number.

cross ratio. (of four collinear points p_1, p_2, p_3 and p_4) the ratio $c(p_1, p_2; p_3, p_4) = (p_{13}/p_{14})/(p_{23}/p_{24})$. It is an invariant under a linear transformation.

Curie point. the transition temperature where ferromagnetism changes into paramagnetism. For example, when a piece of iron gets too hot it is no longer attracted to a magnet.

d, D

Delaunay tessellation. see Delone tessellation.

Delone tessellation. decomposes a Euclidean space of m dimensions into simplexes identical with one another through linear transformations.

deltahedra. Polyhedra which has faces all equilateral triangles. There are eight convex deltahedra, namely tetrahedron, octahedron, icosahedron, triangular dipyramid, pentagonal dipyramid, heccaidecadeltahedron, tetracaidecadeltahedron, and dodecadeltahedron.

δ-slice. a portion of E^d, $d \geq 2$ contained between two hyperplanes orthogonal to a coordinate axis and at a distance of 2δ apart.

deltohedron. solids obtained by twisting one cone of the two cones in regular dipyramids by $1/(2n)$ turn. Then the result is called an n-gonal deltohedron and the original polyhedron a regular n-gonal dipyramid.

depth. (*O.E.* deop)(of a point p in a set S) the number of convex hulls or convex layers that have to be stripped from S before p is removed; (of a set S) the depth of its deepest point.

dihedral angle. the angle created by two intersecting planes.

n^p-distribution. distributions whose expected number of extreme points in a sample of size n is $\mathcal{O}(n^p)$.

dominate. (of a point) having coordinate components in all dimensions greater than another.

e, E

ϵ_0. the permittivity of free space, $\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}.

edge. (*Ger.* die Kante, -n; *Lat.* acies) the arc joining two vertices or a 1-face of a d-dimensional polytope P. ~**-to-**~, (of a tiling of polygons) having all sides and edges coincides, as well as corners and vertices.

elongated. (of polyhedra) having a largest m-sided polygon replaced by an m-prism.

endpoint. (of edges) a vertex.

equiaffine. a subgroup of affine group whose $|A| = \pm 1$, the invariant of which is the volume; Euclidean distance. distance between two points represented as vectors x_i and x_j, that is $|x_i - x_j| = \sqrt{(x_i - x_j)^{\mathrm{T}}(x_i - x_j)} = \left[\sum_{k=1}^{n}(x_{ik} - x_{jk})^2\right]^{1/2}$ for n dimensions.

Euclidean space. a Cartesian space with the Euclidean distance of any dimension.

Euler line. the line containing the circumcentre O, the median point M, the orthocentre H and the centre N of the medial circle. The length of this line is $|OH|$. Then we have $|OM| = (1/3)|OH|$ and $|ON| = |NH|$.

expected complexity. estimate of the average behaviour of an algorithm.

exterior angle. (of a transversal) each of the two angles formed by a halfline, a vertex, and the halfline of a transversal on the side away from the line segment that contains both vertices; (of a triangle) an angle adjacent and supplementary to an [interior] angle of a triangle.

extreme point. (of a convex set) a point $p \in S$ convex set where $\nexists a, b \in S$ such that p lies on the open line segment $\overline{ab}$;

f, F

face. (*Ger.* die Fläche; *Lat.* hedrae).

facet. a $(d-1)$-face of a polytope in d dimensions. see also *subfacet.*

far. (*O.E.* feorr) **vertex of a ~ type**, is when $|T \bigoplus T_1 \bigoplus T_2| = k - 2$, if $k = 1$.

flux. the volumetric flow per cross sectional area, $j = \mathrm{d}V/(A\mathrm{d}t)$.

full period generator. a linear congruential generator whose period is k.

g, G

galactic. *adj* Pertaining or belonging to the Milky Way Galaxy.

galaxian. *adj* Pertaining or belonging to a galaxy.

n-gon. a polygon with n sides and n corners.

gyroelongated. (of polyhedra) having one largest m-sided polygon replaced by an m-gonal antiprism.

h, H

half-plane. (of a plane P with respect to a line m relative to a point $A \in P, A \notin m$) $P_1 = \{A\} \cup \{X | X \in P, X \notin m, (A, X) \cap m = \emptyset\}$ and $P_2 = \{X | X \in P, X \notin m, , (A, X) \cap m \neq \emptyset\}$.

half-space. the portion of E^d lying on one side of a hyperplane. Or if P is a plane and A a point not on P, then the two halfspaces with respect to P relative to A are $\S_1 = \{A\} \cup \{X | X \in P, (A, X) \cap P = \emptyset\}$ and $\S_2 = \{X | X \in P, (A, X) \cap P \neq \emptyset\}$.

n-hedral tiling. a tiling with n distinct prototiles.

homeomorphism. topological equivalence.

homogeneous coordinates. coordinates obtained from projection of points from the inhomogeneous or conventional coordinates represented by the hyperplane $x_{d+1} = 1$ onto the unit hemisphere S^{d+1} of E^{d+1} represents the points at infinity by letting $\varepsilon_{d+1} = 0$.

hyperboulia. excess of will, urgency.

hyperkinesia. increased speed, violence, force, and spread of movement.

hyperplane. a vector space of codimension 1. A hyperplane in an n-dimensional hyperspace is a linear space of $(n-1)$ dimensions. It separates the hyperspace into three parts, *viz.* itself and two other parts which are homeomorphic to the original space.

i, I

incentre. the point where the medians of a triangle intersect.

incidence. the membership of a point p on a line l, is an invariant in affine geometry.

incircle. (of a triangle) the circle which has the incentre of that triangle as its centre and touches all three sides of the triangle.

independence. (of a axiom) cannot be proved by using one or more other axioms within of the same set.

inscribed circle. see *incircle.*

interior angle. (made by a transversal) an angle made by the halfline transversal on the side which contains the segment between the two vertices, a vertex, and one halfline of the other line attached to that vertex.

inversion. (in E^d) a point-to-point transformation of E^d which maps a vector v applied to the origin to the vector $v' = v1/|v|^2$.

isometry. *n* a distance-preserving mapping, a synonym for congruence transformation, comprises of rotation, translation, reflection and the latter two combined which is called a glide reflection; a transformation in which a figure and its image are equal reflections of each other.

isomorphic. one to one. ~ **models**, (in an axiomatic system S) those models in which there exists at least one relation-preserving, one-to-one correspondence between each of their elements, in other words every true statement made about elements in one set is also true about the corresponding elements in the other set.

isomorphism. a one to one and onto relation.

isosceles triangle. a triangle which has two equal sides, *i.e.* where two of the sides are congruent.

isotone. monotone nonincreasing or nondecreasing.

k, K

kagome. *Jpn.* a basket pattern.

l, L

Leech lattice. the lattice in 24 dimensions which has the maximum packing number.

linear. (*Lat.* linearis) ~ **combination**, $p = \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k$ for $p_i \in E^d, \alpha_i \in \Re, i = 1, 2, \ldots, k$. ~ **congruential generator**, a generator which produces random numbers $R_i \in [0, 1)$, where $R_i = X_i/k$, $X_i \in [0, k-1]$, $X_{i+1} = (sX_i + c) \bmod k$, $i = 1, 2, \ldots$, $X_0, m, c, k \in \mathcal{I}^+$, X_0 is the *seed*, m the *multiplier*, c the *increment* and k the *modulus.* ~ **set**, an affine set which passes through the origin.

link. a branch that is not a part of a tree.

lune. a combination where two triangles are attached to opposite sides of a square.

m, M

mapping. association of each preimage point in the domain subset of the source set to exactly one image point in the range subset of the target set. **one-to-one** ~, a mapping where there is no two different images with the same preimage. **onto** ~, a mapping where the range is the union of all subsets of images

medial circle. the circle which passes through the mid points of the sides of a triangle, the feet of its altitudes and the mid of the lines from the point where the altitudes intersect to the corresponding vertices. It is also known as the nine-points circle or the pedal circle because of these nine points it passes through.

median. (of a triangle) bisector of any one of its three angles. also ~ **line**,.

mesh. a closed loop in a graph. **basic** ~, a closed loop formed from the tree by one link of the graph.

metric. a generalized distance.

Möbius strips. An n^{th}-order Möbius strip is a band obtained by joining the two ends of a rectangular strip into a loop after having twisted one of them by an angle $n\pi$, where n is an integer. When the strip is cut along the centre-line, if n is odd the result is one strip having $2n + 2$ half twists which is knotted when $n \geq 3$. If n is even the result is two strips.

monohedral tilings. tilings in which every tile congruent with one another.

monotone. (*Fr.* monotonie; *Gr.* monotonos) ~ **chain**, (with respect to a straight line l) a chain which is intersected by a line orthogonal to l exactly one point. ~ **polygon**, (with respect to a straight line l) a simple polygon whose boundary is the union of two chains monotone with respect to l.

mylonite. Laminated rock with fine grain which is the product of grinding or granulation within the tectonic fault zones.

Monte Carlo. method first done during the 1940s, involve partial differential and integral equations and multi-dimensional integrals, stereotypically maps a deterministic system onto a sampling experiment, from which are collected random samples whose results of statistical analysis give an estimate solution to the problem in real system.

n, N

neighbour. that which has a certain thing in common, for example a common vertex or a common edge.

nine-points circle. see medial circle.

o, O

obtuse. (*Lat.* obtusus) ~ **angle**, an angle which is greater than a right angle, or one which is neither a right- nor an acute angle.

open ball. $N_\epsilon(c) = \{x \mid |x - c| < \epsilon\}$ for $c \in R^m, \epsilon > 0$.

order. (*O.Fr.* ordre) ~ **of**, (implies that) multiplication by constants is involved. ~ **of connectedness**, (of a graph) the maximum number of edges which can be removed without changing the number of components. ~ **of a Voronoi diagram**, $k = |T|$, $\mathrm{V}_{n_k}(S) = \bigcup_i V(T)$, $T \subset S$.

orthocentre. (of a triangle) the point where the lines drawn from the vertices normal to the respective oppositesides intersect, the point of concurrency of its three altitude lines.

orthogonal. (*Gr.* orthos; gonia) right angle. ~ **group**, intersection between the similarity and the equiaffine groups, an invariant of which is the distance. ~ **vectors**, those vectors which are at right angle to each other; $|x_1 + x_2| = |x_1 + x_2|$, $(x_1 + x_2)^2 = (x_1 + x_2)^2$, $(x_1 + x_2)^{\mathrm{T}}(x_1 + x_2) = (x_1 - x_2)^{\mathrm{T}}(x_1 - x_2)$, $x_1^{\mathrm{T}} x_2 = 0$. ~ **projection**, a set of orthogonal projections of point.

orthographic. (*Gr.* orthos; graphein; *Lat.* orthographia) ~ **projection**, a set of orthogonal projections of point.

p, P

packing. a family of non-overlapping sets in a plane.

parametral plane. (in crystallography) the plane which cuts all the three axes of a crystal.

partition. (of S) each of the two or more nonempty and disjoint subsets of S.

path. a self-avoiding path, ie a path whose all vertices are distinct.

pencil. (*Lat.* penicillus): ~ **of lines**, a set of concurrent lines.

perseveration. an uncontrollable self-stimulating and self-maintaining which causes the indefinite continuation or repetition of nervous processes.

polyhedral. (*Gr.* polyedros) ~ **set**, the intersection of a finite set of closed half-spaces in E^d.

polytope. (also d-polytope) convex d-polytope or a bounded d-dimensional polyhedral set.

projective. (*Lat.* projectum) ~ **group**, a full linear group on $d+1$ homogeneous coordinates where $|B| = [A, 0; c, 1] \neq 0$ in $(x', 1) = (x, 1)B$

r, R

regression. a problem of best approximation in a subspace. ~ **function**, some function f^* of $d-1$ variables which minimizes the norm $\mid f - f^* \mid$, where f is a function of $d-1$ variables which represents a set of points in E^d.

regular. (*Lat.* regula) ~ **polygon**, a polygon with equal sides and equal angles. ~ **polytope**, (or a regular polyhedron) a polytope with all faces congruent regular polygons. There are only five distinct types of these and they are called the Platonic solids; (of a vertex) v_j, when there are $i < j < k$ such that (v_i, v_j) and (v_j, v_k) are edges of a graph G whose vertices are indexed in such a way that $i < j$ means either $y_i < y_j$ or, $y_i = y_j$ and $x_i > x_j$.

remote. (*Lat.* removere, remotus) ~ **exterior angle**, (of an interior angle) an angle not adjacent to the interior angle. ~ **interior angle**, (of an exterior angle) an angle not adjacent to the exterior angle.

rheology. the study of the flow of matter.

ridge. boundary element of a facet.

rigid. (*Lat.* rigere, rigidus) ~ **motions**, affine transformations which preserve distance, which are the essence of Euclidean geometry.

route. a path with possible double points.

s, S

side. edge. A side is to a corner what an edge is to a vertex.

similarity. (*Fr.* similaire; *Lat.* similis) ~ **group**, an affine group which has $AA^{\mathrm{T}} = \lambda^2 I$, the ratio of distances between points are preserved.

simple. (*O.Fr.* simple; *Lat.* simplus) ~ ***d*-polytope**, a polytope whose vertices meet exactly d edges. A simple polytope is a dual of a simplicial one.

simlex. a convex hull. **Euclidean** ~, ***d*-simplex**, a d-polytope P which is the convex hull of $(d-1)$ affinely independent points. It contains the total number of 2^{k+1} of k-faces where $k \in \mathcal{I}\, k \geq -1$, an empty set being $k = -1$. A simplex for $d = 0$ is a vertex, for $d = 1$ an edge, for $d = 2$ a triangle and for $d = 3$ a tetrahedron.

simplicial. (*Lat.* simplex, simplicis) ~ ***d*-polytope**, a polytope all the facets of which are simplices.

simply connected. is connected and contains no holes.

site. a vertex, to be distinguished from a nucleus or generator of Voronoi networks which is also a site but of the dual network.

space. (*O.Fr.* espace; *Lat.* spatium) ~ **points**, at least four points which are not necessarily coplanar.

sparsity. (*Lat.* sparsus) measure of sparseness in point distribution, a point set $S \in E^d$ has sparsity $c \in \mathcal{I} \geq 1$ for a given $\delta \in \mathcal{R}^+$ iff there are at most c points of S within any box or hypercube of side 2δ. In other words, sparsity is the scarcity of points within a given box. It is preserved through orthogonal projection.

spheno-. *prf* a wedgelike combination formed by two adjacent lunes. **di~-**, two such combinations. **hebe~-**, two lunes separated by a third one.

snub. (of polyhedra) *adj* resulting from a chiral process of rotating all faces of a polyhedron in the same direction. This creates one m-sided polygon for each vertex of degree m and two triangles for each edge. A polyhedron has the same snub as its dual.

stereohedra. set of regions whose congruent copies fill three dimensional space without overlap except at their boundaries.

stochastic. (*Gr.* stokhastikos) ~ **systems**, physical systems which involve random process evolving over time.

subfacets. the $(d-2)$-faces of a polytope P in d dimensions.

t, T

temporomandibular jaw joint disease. The condition of painful jaw joint, sometimes also ears, neck, shoulders and back. Possible symptoms include uncomfortability when openning the mouth and clicking sound when moving the jaw joint.

tessellated. (*Lat.* tessera, tessella; *Gr.* tessares) ~ **polyhedra**, polyhedral cells having the same number of faces.

tessellation. (from *Lat. tessellātus.* mosaic) a space comprised of tiles or simplexes identical to one another via linear transformation; space entirely covered with a pattern.

tesseract. a four-dimensional hypercube.

tile. each piece of, or each set in, a tiling.

tiling. (*Lat.* tegere, tegula; *O.E.* tigele) **plane** ~, a countable family of closed sets which covers a plane and leaves no gap.

tinnitus. The perception of clicking, hissing, popping, ringing, rumbling, or other sound in the ears when no external sound is present. Causes are numerous and include bad positioning of the neck and atherosclerosis. Suggested treatment ranges from taking vitamin A, E, magnesium, potassium and zinc to chiropractice.

tree. a set of branches connecting all the nodes of the graph without forming any closed loops or meshes.

truncated. (of polyhedra) having an k-gonal pyramid cut off from one or more of the vertices.

v, V

valence. (*Lat.* valere, valentia) n-~, having a vertex which is an end point of n edges.

vertex. *pl.* vertices. (*Ger.* die Ecke, -n; *Lat.* angulorum solidum) a point in any dimension. When creating a Voronoi diagram the nuclei are vertices of the corresponding Delaunay diagram, while a Voronoi vertex is the circumcentre of a facet of the Delaunay triangulation of a convex hull one dimension higher. ~ **of a pencil**, the point through which all lines in a pencil pass. Vertices are many a vertex or isolated points connected to edges, they are 0-faces of a polytope P in d dimensions.

Voronoi. *n* George Fedosevich Voronoi; Voronoi tessellation, Voronoi networks, *etc.* **generalised ~ diagram**, $V(T) = \{p : \forall v \in T, \forall w \in (S-T), d(p,v) < d(p,w)\}$ or $V(T) = \bigcap_{ij} H(p_i,p_j), p_i \in T, p_j \in (S-T)$, where $H(p_i,p_j)$ is the half-plane containing p_i that which is defined by the perpendicular bisector of $\overline{p_ip_j}$. ~ **diagram**, (also called area of influence polygons (mining), area potentially available to a tree (forestry), capillary domains, Dirichlet tessellation, domain of an atom (metallurgy), plant polygons (ecology), plesiohedra (one kind of stereohedra), Thiessen polygon, Wigner-Seitz regions (physics), Wirkungsbereich (crystallography)).

worst-case complexity. measures the performance of an algorithm over all problem instances.

§ B.1 Abbreviation

a, A

AIESEC. Association Internationale des Etudiants en Sciences Economiques et Commerciales.

ATPIJ. The Association of Thai Professionals in Japan.

ATSIST. The Association of Thai Students in Science and Technology Professions.

c, C

cf. *Lat.* `confer`, compare.

c.g.. centre of gravity.

d, D

DT. Delaunay triangulation; Dirichlet tessellation.

e, E

eg. *Lat.* `exempli gratia`, for example.

i, I

IEEE. Institute of Electrical and Electronics Engineers, Inc..

IUPAC. International Union of Pure and Applied Chemistry. `www.iupac.org`.

p, P

Pc. percolation. ~-**process**, percolation process.

p.d.f.. probability density function.

PM. porous media; porous membranes.

s, S

SEM. scanning electron microscope.

s.t.. such that.

STP. standard temperature and pressure.

t, T

TC. twentieth century.

TEM. transmission electron microscope.

TIT. Tokyo Institute of Technology.

u, U

UMIST. University of Manchester Institute of Science and Technology.

v, V

VP. Voronoi Percolation; the study of percolation on Voronoi networks; any application of Voronoi tessellation in the percolation theory.

VT. Voronoi tessellation.

§ B.2 Biographies

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi. b. *circa* 780, Baghdad; d. *circa* 850.

Lidwig van Beethoven. b. 17th December 1770 , Bonn; d. 26th March 1827 , Vienna. An accomplished pianist and composer who wrote most wonderful piano sonatas and helped shape the

Romantic period of classical music. He began to have problems with his hearing facility in 1796 which developed to become a total deafness which most modern otologists decide are caused by otosclerosis of the mixed type.

Brahmagupta. b. 598, Ujjain, India; d. *circa* 670, India.

Jean Le Rond d'Alembert. b. 17th November 1717 , Paris, France; d. 29th October 1783 , Paris, France.

Girolamo Cardano. b. 24th September 1501 , Pavia, Duchy of Milan; d. 21st September 1576 , Rome.

Augustin Louis Cauchy. b. 21st August 1789 , Paris, France; d. 23rd May 1857 , Sceaux, France.

Arthur Cayley. b. 16th August 1821 , Richmond, Surrey, England; d. 26th January 1895 , Cambridge, Cambridgeshire, England. Another mathematician from a Yorkshire family who lived in Cambridge. He contributed to matrices, non-Euclidean geometry and the abstract group concept.

Edward Salisbury Dana. b. 16th November 1849 , New Haven, Connecticut; d. 1935. Studied at Yale, where he received his Ph.D. in 1876, as well as in Heidelberg and Vienna, he was the son of James Dwight Dana who also wrote books as well as appendices to his father's *System of Mineralogy.*

James Dwight Dana. b. 12th February 1813 , Utica, New York; d. 4th April 1895 . Studied at Yale and joined the navy, he taught at Yale and married Henrietta Frances, the third daughter of Professor Benjamin Silliman whom he assisted there. He wrote a most definitive Manual of Geology, received a Ph.D. from the University of Munich on its fourth centennial celebration in 1872, and continued working very hard into the last year of his life.

Henry Philibert Gaspard Darcy. b. 30th June 1803 , Dijon, Départment de la Côte d'Or, France; d. 2nd January 1858 , Paris. Discovered the Darcy's law of flow in porous media. Invented the modern style Pitot tube. Noticed the existence of the boundary layer in fluid flow. His name is sometimes wrongly written 'D'Arcy'. This has been verified as his school photo in 1821 already wrote the name as 'Darcy'. It is interesting that his name should have always been written in such an anglicised way as 'Henry', whereas the name of his wife, on the other hand who was originally English, used a french spelling, 'Henriette Carey'.

Jean Baptiste Louis Romé Delisle. b. 1736, Gray, eastern France; d. 1790, Paris.

Boris Nikolaevich Delone. b. 15th March 1890 , St Petersburg, Russia; d. 1980.

René Descartes. b. 31st March 1596 , La Haye, Touraine, France; d. 11th February 1650 , Stockholm, Sweden. In 1647 he met Pascal in France and argued with him that a vacuum could not exist. The curve of the equation $x^3 + y^3 = 3axy$ which he discussed in 1638 is now called the Folium of Descartes, though it is no longer associated with flowers' petals. He solved $x^2 + ax = b^2$ with ruler and compass by writing it as $(x + a/2)^2 = (a/2)^2 + b^2$ and seeing that x is nothing but the distance from the corner A to the circle, centred at B, which has a diameter of a and touches a line segment of length b, $\overline{AC}$, at C.

Abraham de Moivre. b. 26th May 1667 , Vitry near Paris, France; d. 27th November 1754 , London, England.

Johann Peter Gustav Lejeune Dirichlet. b. 13th February 1805 , Düren, French Empire; d. 5th May 1859 , Göttingen, Hanover. The young from Richelet or *Le jeune de Richelet*, for the town in Belgium where his family came from, he is not from France as many had claimed. In his youth he showed interests in history and mathematics. He treasured his copy of Gauss's Disquisitiones arithmeticae as others might a bible. When Gauss died in 1855, he was offered his chair at Göttingen. With him the golden age of mathematics in Berlin began. His proofs are characterised by surprisingly simple initial observations followed by extremely sharp analysis of the problem.

Diophantus of Alexandria. b. *circa* 200; d. *circa* 284.

Johann Peter Gustav Lejeune Dirichlet. His name originated from *Le jeune de Richelet* [The young from Richelet, a town in Belgium where his family came from.]

Euclid of Alexandria. b. *circa* 325 BC; d. *circa* 265 BC, Alexandria, Egypt.

Leonhard Euler. b. 15th April 1707 , Basel, Switzerland; d. 18th September 1783 , St. Petersburg, Russia.

Edward Morgan Forster. b. 1879; d. 1970.

Johann Carl Friedrich Gauss. b. 30^{th} April 1777 , Brunswick, Duchy of Brunswick; d. 23^{rd} February 1855 , Göttingen, Hanover. At seven he discovered that $\sum_{i=1}^{1} 00i = 50 \times 101$. His doctoral dissertation was a discussion of the fundamental theorem of algebra. He is interested in differential geometry, where he discovered that the Gaussian curvature is invariant under isometric transformations of area in E^3. He is also interested in magnetism and worked with Weber. Together they discovered the Kirchoff's theory.

William Rowan Hamilton. b. 3^{rd} or 4^{th} August, 1805; d. 1865. When a child, he was taught 14 languages, and at 17 taught himself mathematics and there by discovered an error in Laplace's Celestial Mechanics. He is credited for having invented the quaternions, and sometimes for having scratched the result of that discovery, that is $i^2 = j^2 = k^2 = ijk = 1$, on the stone of the Brougham bridge on the Royal Canal. He also invented the icosian game where one is asked to find a path along a polyhedron's edges such that each node is visited only and at least once.

Abraham bar Hiyya Ha-Nasi. b. 1070, Barcelona, Spain; d. 1136, Provence, France.

Henry Selby Hele-Shaw. b. 1854, Billericay, Essex; d. 1941. He taught at University College, Liverpool. He was elected to the Royal Society in 1899 because of the fundamental investigation he had carried out regarding streamline flow of liquids.

Charles Hermite. b. 24^{th} December 1822 , Dieuze, Lorraine, France; d. 14^{th} January 1901 , Paris, France.

Joseph-Louis Lagrange. b. 25^{th} January 1736 , Turin, Sardinia-Piedmont; d. 10^{th} April 1813 , Paris, France.

Pierre Laplace. b. 1749; d. 1827. It is often said that his five volume *Mécanique Céleste* (1799–1825) is in great part a summation of works by his predecessors, as a result of which he often omitted derivations by writing them off as being obvious and easy to see.

Sir Joseph Larmor. b. 11^{th} July 1857 , Magheragall, County Antrim, Ireland; d. 19^{th} May 1942 , Holywood, County Down, Ireland. He was a Lucasian Professor of Mathematics at Cambridge from 1903 until 1932 when he was succeeded by Dirac.

Gottfried Wilhelm von Leibniz. b. 1^{st} July 1646 , Leipzig, Saxony; d. 14^{th} November 1716 , Hannover, Hanover.

Hendrik Antoon Lorentz. b. 18^{th} July 1853 , Arnhem, Netherlands; d. 4^{th} February 1928 Haarlem, Netherlands.

William Hallowes Miller. b. 1801, Velinde, near Llandovery, South Wales; d. 1880, Cambridge.

Hermann Minkowski. b. 22^{nd} June 1864 , Alexotas, Russian Empire, now Kaunas, Lithuania; d. 12^{th} January 1909 Göttingen, Germany.

Franz Ernst Neumann. b. 1798, Joachimsthal; d. 1895.

Luca Pacioli. b. 1445, Sansepolcro, Italy; d. 1517, Sansepolcro, Italy. His *Summa de arithmetica geometria, proportioni et proportionalita*, published in 1494, summarises the contemporary algebra, arithmetic, geometry, and trigonometry. In 1509 published the *Divina proportione* which deals with the golden ratio and contains illustrations by Leonardo da Vinci.

Parmenides. b. *circa* 515; d. after 450 BC. He reasoned that since a void is nothingness, if two particles were separated by a void, then they would be separated by nothing. In other words, they would not be separated at all, they would be touching (*cf* Davies, 2001).

Jean Louis Poisseuille. b. 1799; d. 1869.

Simeon-Denis Poisson. b. 1781; d. 1840.

Scipione del Ferro. b. 6^{th} February 1465 , Bologna, Italy; d. 5^{th} November 1526 , Bologna, Italy.

Niels Stensen (aka Nicolaus Steno). b. 1638, Copenhagen; d. 1686, Schwerin.

Robert Louis Stevenson. (1850–1894)

James Stirling. b. May 1692, Garden near Stirling, Scotland; d. 5^{th} December 1770 , Edinburgh, Scotland.

Johannes Diderik van der Waals. b. 23^{rd} November 1837 , Leyden, The Netherlands; d. 8^{th} March 1923 , Amsterdam. It was him who coined the equation for real gas, $(p + an^2/V^2)(V - nb) = nRT$.

George Fedosevich Voronoi. b. 28^{th} April 1868 , Zhuravka, Poltava guberniya, Russia (now Ukraine); d. 20^{th} November 1908 , Warsaw, Poland. Both his master's degree, 1894, on the algebraic integers associated with the roots of an irreducible cubic equation and his doctoral thesis on algorithms for continued fractions were awarded the Bunyakovsky prize by the St. Petersburg Academy of Sciences. But he decided that he wanted to teach at the Warsaw University where he extended work by Zolotarev on algebraic numbers and the geometry of numbers. He met Minkowski in 1904 at an international conference at Heidelberg.

Egor Ivanovich Zolotarev. b. 12^{th} April 1847 , St Petersburg, Russia; d. 19^{th} July 1878 , St Petersburg, Russia. Received a silver medal from the Gymnasium in St. Petersburg, attended lectures by Kummer and Weierstrass, and discussed mathematics with Hermite, he worked with Korkin and gives complete solutions to the four- and five variable cases of the problem of finding the minimal values of n-variable quadratic forms with real coefficient.

§ B.3 Computation and softwares

AVS. *AVS can be used to find cross sections in 2– and 3–D networks.*

hull. *Hull is written in ANSI C by Ken Clarkson. It computes the convex hull of a point set of any dimension.*

Synopsis: hull -d -f<format> -A -aa<alpha> -af<format> -oN -ov -s<seed> -r -m<multiplier> -X<debug file> -i<input file> -oF<output file>

Hull takes in points as its input. The outputs are vertices of the convex hull facets, Delaunay triangulations, alpha shapes, and volumes; in postcript or OFF format for geomview.

hullio.a precursor to *hull.*

Matlab. *Matlab has an algorithm for finding 2–D voronoi diagrams. However, the output data for this is not very structured, which makes it difficult to use the data obtained for analysis purpose.*

When repeatedly running a .m file online, all variables should be cleared by 'clear all' two times, both at the beginning and at the end of the file. Failing to do so sometimes results in consistency in the results.

Submitting .m files through NQS often requires writing every path in full. However, in .m files one can write 'path(path,*directory*)' or 'path(*directory*, path)' for post-appending and pre-appending a path, for example that which contains the .m files containing functions.

I guess that everything done on a matrix in Matlab is as fully vectorised as possible, since provided that this is the case that very program can still be greatly improved, and I hold the programmers who develop it in a better regard than what would have allowed me to assume this. So we can vectorise our algorithm by simply putting the various items into a matrix and work on them, in that matrix, in parallel instead of in sequence as we would normally do. There is still a limitation in that we can subject a matrix to only one operation, and therefore can only do in parallel things which require the same operation.

Another limitation is that we can only put matrices into a structure, which in Matlab contains data of the class *cell*, but not vice versa. So the great convenience we have from working with the cell structure comes at the cost to parallelisation. In effect, this means that we can only parallelise our algorithmic details but not the whole algorithm itself. In other words, we can not simultaneously find the percolation probability for two different networks using Matlab.

qhull. *A quick hull C program for finding convex hulls, Delaunay triangulations, Voronoi vertices, furthest-site Voronoi vertices, halfspace intersection about a point, hull volume and facet area. It is written by C. Bradford Barber and Hannu Huhdanpaa at The Geometry Center, University of Minnesota (Barber et al, 1996). The program combines the 2-d Quickhull algorithm with the general dimension Beneath-Beyond algorithm. The latter is an incremental algorithm which adds a point to the convex hull of the points just processed. It processes a new point in steps as described in Algorithm 9.1. The boundary of the visible facets is the set of horizontal ridges for the point. A* *facet is visible to a point if the latter is above it. Cones of new facets are constructed from the point of its horizon ridges.*

Algorithm 9.1 *Beneath-Beyond algorithm*

```
for [each new point] do
    locate facets visible to it;
    construct a cone of new facets;
    delete the visible facets;
endfor
```

□

There are various kinds of quickhull algorithms. The one mentioned here is the work of Barber *et al* (*ibid.*) which works in the space of points and convex hulls and maintains an outside set for each facet. Being in the outside set implies that a point is above the facet. It is one of the variations of the randomised incremental algorithm proposed by Clarkson and Shor (1989) which works in the space of halfspaces and polytopes, dual to that of the present one, and maintains a conflict graph, a set of all the list of polytope edges that intersect an unprocessed halfspace. Some of the options are *d* Delaunay triangulation by lifting points to a paraboloid,

d Qu computes the furthest-site Delaunay triangulation from the upper convex hull, *f* print all fields of all facets, *FA* computes total area and volume, *FN* lists the voronoi vertices for each voronoi region, *Fp* halfspace intersection coordinates (*F* for output format), *Ft* prints a triangulation (with points (the centrums) added to non-simplicial facets), *Fv* when used with *v* option prints a (furthest-site) Voronoi diagram (Output: number of ridges \\ `< the count of indices> <1st input site> <2nd input site> <1st ridge> <2nd ridge> ... \\ ...`), *Fx* convex hull vertices, *G* Geomview output (2– to 4–*d*), *H*n,n,... computes halfspace intersection about $[n, n, 0, \ldots]$ (The point $[n, n, n, \ldots]$ lies inside $Hx + b \leq 0$, default $b = 0$), *m* Mathematica output (2– and 3–*d*, in Mathematica <*variable*>, <<, <*filename*> then *Show[Graphics3D[list]]*), *o* prints the input points and facets, *Pg* prints only good facets, *QV*n a good facet includes point *n* ($n < 0$ means a good facet does not include point *n*), *Qg* builds good facets, *TO ¡file¿*, *Tv* verify structure, convexity, and point inclusion of the result, *p* vertex coordinates, *i* vertices incident to each facet, *P* printing, *Q* qhull control, *Qb*k:n scales the k^{th} coordinate of the input points (the lower bound of the input points becomes *n*)*Qb*k:*0B*k:*0* drops dimension *k* from the input points before the Delaunay and Voronoi to allows sub-dimensional convex hulls, *QB*k:n the upper bound becomes *n*, *Qbb* scales the last coordinate to $[0, m]$ where *m* is the maximum absolute value of the other coordinates,*QbB* scales the input points to fit the unit cube after projection to the paraboloid. The lower and the upper bounds for all dimensions are -0.5 and $+0.5$, *QJ* triangular output, *Fa* prints area for each facet, *PA*n prints the *n* largest facets, *PF*n prints facets larger than *n*, *Fs* prints a summary of the structure, *s* prints a summary to stderr, *T* tracing, *v* Voronoi diagram via the Delaunay triangulation, *v Qu* finds the furthest-site Voronoi diagram,

Examples:

```
rbox <number of points> t<seed> D3 | qhull QV5 v p Pg | qhull G > <filename>;
rbox <number of points> t<seed> D3 | qhull v Fs > <filename>;
```

A *d*-d convex hull in this algorithm is represented by its vertices and $(d-1)$-d facets or faces. Extreme points are those which are the vertices of a convex hull. Each facet has a set of vertices, a set of neighbouring facets and a hyperplane equation. A $(d-2)$-d face is a ridge of the convex hull, which is the intersection of the vertices of two neighbouring facets. An oriented hyperplane through *d* points is represented by its unit normal which points outwards and its offset from the origin. The signed distance from a point to a hyperplane is the inner product of the point and the normal plus the offset. A hyperplane defines a halfspace of points having negative distance from it. The point is above the hyperplane if this distance is positive.

In R^d, Quickhull repairs in order faults where more than two facets meet at a ridge, a facet is in another facet, a facet has fewer than *d* neighbours, a facet has a flipped orientation, a point just processed is coplanar with a horizontal facet, concave facets, coplanar facets and redundant vertices.

The program *rbox* is written in C. It generates pseudo random points for *qhull.* The arguments *t*n tells it to use *n* as a random seed. Here *D2* means 2-dimension while *D3* 3-dimension.

rbox. *This program generates random points for qhull. When used without an option gives a list of possible options.*

Sweep2. *This is a program for creating 2-dimensional Voronoi diagrams and Delaunay triangulation. It uses sweepline algorithm and is written by Steve Fortune.*

triangle. *A C program by Jonathan Richard Shewchuk for 2–D triangulation and mesh generation. It uses Ruppert's Delaunay refinement algorithm.*

volume. *Written by Joseph O'Rourke, it finds the volume of a simple polyhedron from the triangulated surface input read from stdin. Inputs are vertex coordinates represented as integers and triangle faces as vertex indices.*

§ B.4 Internet resources

`www.faqs.org` *Internet FAQ Consortium.* A site for Frequently Asked Questions which covers various areas. Useful algorithms for geometrical computation can be found here together with their references.

`www.geom.umn.edu` *Geometry Center.* Being at the University of Minnesota, this is the place to find the programs Quickhull (Qhull) and Rbox used in the beginning of this study. Qhull is incorporated into MATLAB as the commands Qhull, Convhulln, Delaunayn and Delaunay3. It is written by C. Bradford Barber and Hannu Huhdanpaa.

`www.gnu.org` *The "GNU's not Unix!" Project and Free Software Foundation.* Recursively named, this project originated by Richard Stallman offers a variety of profound softwares, most of which are for Unix and Linux developed by tinkerers. Most of the softwares come with codes, therefore are ideal for developpers. The 3-d viewing program called Geomview is only one example.

`www-groups.dcs.st-and.ac.uk` *Turnbull Server.* Named after Herbert Westren Turnbull (1885–1961), this server is valuable to anyone who has an interest in Mathematics, as well as researchers in history of this field. It houses the MacTutor History of Mathematics archive, which covers history of the subject and biographies of mathematicians. The materials offered are extensive.

`www.gutenberg.org` *Gutenberg Project.* A site containing valuable books on many topics, including but not only literatures.

`www.mathworks.co.uk` *MathWorks Developers of MATLAB and Simulink.* A useful site for users of both products. One of the strong points of products of MathWorks is the extensiveness of their help facilities. The helps available here are better organised and more explanatory than those that come with the program.

`www.nectec.or.th` *National Electronics and Computer Technology Center, Thailand.* Supposedly the only research authority in Thailand, this site hosts various other sites of thai researchers all over the world. That of ATSIST, www.nectec.or.th/bureaux/atsist, is but one of them.

§ 9.5 TeXnicalities

LaTeX is a macro which runs on TeX. It gives one convenience but not without a tradeoff in understanding. Also one may not have much freedom in writing macros on LaTeX. Tiyapan used TeX for his Work Notes dated 12th February 2001 .

LaTeX(Lamport, 1985) is written by Leslie Lamport. Newer versions of it has come up at a regular interval. Unlike most other macros which run on TeX, her source code is free for none but herself. Moreover, having used TeX to do what it wants, LaTeX thereby castrates her progenitor in such a way that it is impossible for her users to define new macros efficiently by using the `\def` command. With `\def` disabled, the lion has lost its fangs and users become as docile and dependent as a lobotomised patient. There can be no doubt that with the TeX users having such idea as this, sooner or later LaTeX will have to change in these respects. But this is the way things are at present.

One of the first macros written is the code to change the date format. The algorithm first sets $x = date$, then it assigns the ordinate endings *st, nd, rd* or *th* depending on the value of *ordinate*, which is calculated from

```
if $x > 30$ then $ordinate = 1$ else
  if $x > 20$ then $ordinate = x - 20$ else
    $ordinate = date$
  endif
endif
```

The macros which are either newly written or adapted from elsewhere, mainly from the *manmac* macroby Knuth, are listed in § 9.5. Apart from these, this thesis uses the *plain* and the *manmac* macros. Another set of macros developed here is that which deals with languages. The definition of language here is quite wide. It contains many languages among which are those which are used here, for example the languages for Chemistry, Chinese, Czech, German, French, grammatical jargons, Japanese, Lanna, Latin, latin grammar, Mathematics, Pali, Physics, Russian, Sanskrit, and Thai

(Daiy), *etc.* Only parts of this set of macros are useful for the writing of this thesis, not least so those which are used for writing the dedication page.

In the original account of his, Tiyapan (2003, KNT8(iii)) wrote,

> When I first started using TeX instead of LaTeX, I only used the macro `plain.tex`. Then in my first book typeset with TeX (Tiyapan, 2001, KNT8(ii)), I used in addition to the plain TeX `manmac.tex` and `epsf.tex`. Now to my amazement, I have discovered many other excellent macros, for instance `rotate.tex`, and found that I could understand how they work when I read them. This is one of the benefits that comes with talking in TeX instead of, for instance, LaTeX. I also know now the difference between the primitive TeX and plain TeX, and that the latter is only one of the infinitely many possible implementations of TeX. However, since all TeX gurus I know use plain TeX as a basis, there is no reason why I should be too proud to follow the practice. Having said that, my next plan is to improvise on the primitive TeX without any direct reference to the plain TeX macro.

The citation program BibTeX was intended to be used with LaTeX. Karl Berry and Oren Patashnik have written `btxmac.tex` which makes BibTeX usable from plain TeX. But for the present purpose I merely use my own macros, which are much simpler, and do not need BibTeX. Ultimately such database program as BibTeX would have been extremely useful. But I wish to develop something similar to it on my own.

There are still some unsolved problems in the TeX macros, for example the page references which are embedded within groups are sometimes slightly wrong, that is they may appear to be one more or one less than their actual position. Since publishing macros play but a minor part here compared with mathematics and physics, this problem has been systematically minimised and then tolerated. The solution and explanation of it will be dealt with and published elsewhere.

When lines of text appear beside a picture there are macros which make the latter always stay next to the outer rim of the page. These work satisfactorily well, and those cases in the results which appear to say otherwise are in fact the result of some other more primitive macros earlier written.

§ 9.6 Voronoi statistics of earlier simulations

All the mean numbers for each face of the network of 527 Voronoi cells (§ 9.6) are 4.9091, 5.0769, 5.2500, 5.1429, 4.8000, 4.9091, 5.2500, 5.1429, 5.1429, 5.0769, 5.2000, 5.2000, 5.0000, 5.2500, 4.8000, 5.2941, 5.0769, 5.2000, 4.9091, 5.0769, 4.6667, 5.3684, 5.2000, 5.3333, 4.6667, 5.2500, 5.0000, 4.5000, 5.0000, 5.4286, 5.2000, 5.2500, 5.0769, 5.2500, 5.1429, 4.9091, 5.0000, 5.0000, 5.3333, 4.8000, 5.3333, 5.2000, 5.3684, 5.2500, 5.0000, 5.0769, 5.0769, 5.2941, 4.8000, 5.0000, 5.0769, 5.4286, 5.3333, 5.1429, 4.6667, 4.6667, 5.4783, 5.2941, 5.2500, 5.2500, 4.9091, 4.9091, 5.4000, 5.2500, 5.1429, 5.2941, 5.4545, 5.1429, 5.0000, 4.9091, 5.2500, 5.1429, 5.1429, 5.4545, 4.8000, 5.1429, 5.0000, 5.2941, 5.2941, 5.0000, 5.4286, 5.0769, 5.1429, 5.1429, 5.2941, 5.0769, 4.9091, 5.2000, 5.2000, 5.4545, 5.1429, 5.2500, 5.1429, 5.2000, 5.2500, 5.1429, 5.1429, 5.0000, 5.3684, 5.3684, 5.2941, 5.4000, 5.4783, 5.3684, 4.9091, 5.2941, 5.2500, 4.8000, 5.2500, 5.2941, 5.1429, 5.2500, 5.2500, 5.2000, 5.0769, 5.4000, 5.0000, 5.3333, 5.4286, 4.8000, 5.2500, 5.3333, 5.0769, 4.9091, 5.3333, 5.3333, 5.2941, 5.2000, 5.3333, 5.3333, 5.2500, 5.2500, 5.2500, 5.2000, 5.2000, 5.1429, 5.0000, 4.8000, 5.3684, 5.1429, 5.2000, 5.2500, 5.0769, 5.3684, 5.2941, 5.1429, 5.2500, 5.2000, 5.1429, 4.8000, 5.0000, 5.3333, 5.2000, 5.0769, 5.2000, 5.3333, 4.5000, 5.0000, 5.3333, 5.3333, 5.0000, 5.2500, 5.2000, 4.9091, 5.1429, 5.0000, 5.3684, 5.2500, 5.0000, 5.4000, 5.3333, 5.2500, 4.8000, 5.5000, 5.1429, 5.2000, 5.2500, 5.2941, 5.2500, 5.2941, 5.4000, 5.2941, 5.2941, 4.9091, 5.2000, 5.2000, 5.4286, 5.0769, 5.3684, 5.2500, 5.3333, 5.0000, 4.9091, 5.3684, 4.0000, 5.1429, 5.3333, 4.9091, 5.3333, 5.2000, 5.1429, 5.1429, 5.1429, 5.0000, 5.0769, 5.1429, 5.3684, 4.9091, 5.2941, 5.0000, 5.2941, 5.2941, 5.2500, 5.1429, 5.0769, 5.2000, 5.4000, 5.1429, 5.0769, 5.2000, 4.6667, 5.3684, 5.4286, 5.0769, 5.2500, 5.1429, 5.3684, 5.0769, 5.2500, 5.3684, 5.3684, 5.0769, 4.8000, 5.3333, 5.0769, 5.1429, 5.2000, 4.9091, 5.2000, 4.9091, 5.2500, 5.2000, 4.9091, 5.2500, 5.2941, 5.5000, 5.2000, 5.2500, 5.4783, 5.2941, 5.2000, 5.0000, 5.2000, 5.2941, 5.2000, 5.2941, 4.9091, 5.4286, 5.2500, 5.1429, 5.5000, 5.1429, 5.4286, 5.0000, 5.3684, 5.2500, 5.3684, 5.4000, 5.2500, 5.2941, 4.8000, 4.9091, 4.8000, 5.1429, 5.3333, 5.2000, 5.0000, 5.0769, 5.0000, 5.0769, 5.1429, 5.1429, 5.0769, 5.0769, 5.4545, 5.0769, 5.2000, 5.0769, 5.4545, 5.4286, 4.8000, 5.2000, 5.0769, 5.3684, 5.3333, 5.3333, 5.4000, 5.2000, 5.0000, 5.0000, 5.2941, 5.2000, 4.9091, 5.2500, 5.3333, 5.0000, 5.3684, 5.3333, 5.3333, 5.0000, 5.4545, 5.1429, 5.0769, 5.0769, 5.0000, 5.5200, 5.0769, 5.1429, 5.2000, 5.3333, 5.0000, 5.2941, 5.2500, 5.2500, 5.2500, 5.2000, 5.2000, 5.2000, 5.2000, 5.2000, 5.2000, 5.2000, 5.2500, 5.2500, 4.9091, 5.4000, 5.2941, 5.2000, 5.2500, 5.2500, 5.1429, 5.0769, 5.0769, 5.2000, 5.2941, 5.2941, 5.4783, 5.3333, 4.8000, 5.3333, 5.2941, 5.0769, 5.3333, 5.1429, 5.3684, 5.2941, 5.1429, 5.1429, 5.4545, 5.3333, 4.9091, 5.1429, 5.2941, 5.2500, 5.0769, 5.3333, 4.9091, 5.0769, 5.3333, 5.0000, 4.9091, 5.3333, 5.2941, 5.3684, 5.2941, 5.1429, 5.2500, 5.3333, 5.1429, 5.0769, 5.2000, 5.2500, 5.3333, 5.0769, 5.0000, 5.2941, 4.9091, 5.0769, 5.0769, 5.4545, 5.2000, 5.2941, 4.8000, 5.2941, 5.2500, 5.3684, 5.2000, 4.9091, 5.2000, 5.2000, 5.1429, 4.6667, 5.2000, 5.2500, 4.6667, 5.3333, 5.2941, 5.0769, 5.0769, 5.0000, 5.3333, 5.0000, 5.4000, 5.4545, 5.3333, 5.2500, 5.1429, 5.2000, 4.9091, 5.1429, 5.2000, 5.3333, 5.2941, 5.5000, 5.2500, 5.1429, 5.2000, 5.3684, 5.1429, 5.4286, 5.0769, 5.2941, 5.2941, 5.2500, 5.1429, 5.1429, 5.2500, 5.1429, 5.3333, 4.8000, 5.2000, 5.2000, 4.8000, 5.4286, 5.2000, 5.2941, 5.2500, 5.5000, 5.2941, 5.0000, 5.2941, 5.2500, 4.6667, 5.2941, 5.0000, 5.2500, 5.2500, 4.9091, 5.2000, 5.2941, 5.0769, 5.3684, 4.9091, 4.9091, 5.2000, 5.4000, 5.2000, 5.0769, 5.3684, 5.2000, 5.2500, 5.0769, 5.2500, 5.0000, 5.2941, 4.5000, 5.3333, 5.2000, 5.2941, 5.1429, 5.0000, 5.0769, 4.5000, 5.4783, 5.3684, 5.1429, 4.9091, 5.3684, 5.2000, 5.2500, 5.0000, 5.2941, 5.0000, 5.0769, 5.0000, 5.3333, 5.2000, 5.2000, 5.2500, 5.2000, 5.2000, 5.2000, 4.8000, 5.0769, 5.2000, 5.2941, 5.2941, 5.2500, 5.0000, 5.4286, 5.2000, 5.2500, 5.3333, 5.2000, 5.2000, 5.0000, 5.1429, 5.2000, 5.2941, 5.2941, 5.3333, 4.8000, 5.3333, 4.6667, 5.1429, 5.1429 and 5.2000.

All these standard deviations are 1.4460, 1.5525, 1.5706, 2.3157, 1.2293, 1.5783, 1.5275, 1.0995, 1.0271, 1.7541, 1.4736, 1.2649, 1.5954, 1.2910, 0.6325, 1.5315, 0.9541, 1.2071, 0.7006, 0.8623, 1.0000, 1.3829, 1.5675, 1.7150, 1.2247, 1.6533, 1.5374, 0.9258, 1.1282, 1.8323, 1.9346, 1.3416, 0.6405, 1.8439, 1.6575, 1.3003, 1.4142, 1.5374, 1.5339, 1.1353, 1.5718, 0.8619, 2.1137, 1.7321, 1.5954, 1.6053, 1.3205, 1.7235, 0.7888, 1.2792, 1.7059, 1.6301, 1.4142, 1.6575, 1.3229, 0.8660, 1.9038, 1.9610, 1.9149, 1.8074, 0.9439, 1.5136, 1.9841, 1.3904, 1.6575, 1.2632, 2.0172, 1.7913, 1.5954, 1.9212, 1.4376, 1.9945, 1.6104, 1.5954, 1.1353, 1.8752, 1.4142, 1.7235, 1.4902, 1.4142, 1.5991, 1.3821, 1.8337, 1.0271, 1.6494, 1.0377, 2.0226, 1.6987, 1.9712, 1.2994, 1.5119, 0.9309, 1.9555, 1.0142, 1.0646, 1.2924, 2.1070, 1.8586, 2.0873, 1.4225, 2.2573, 1.7290, 1.7547, 1.9779, 1.6404, 1.7946, 2.2361, 1.4757, 1.6125, 1.4038, 0.9493, 1.6533, 1.5706, 1.6125, 1.8467, 1.4290, 1.0445, 1.7489, 1.5675, 1.3984, 1.1832, 1.9704, 0.9541, 1.3003, 2.1420, 1.6803, 1.6111, 2.2104, 1.9704, 1.5718, 1.2383, 1.3416, 1.6125, 2.1778, 1.5675, 1.6104, 1.1282, 1.3166, 2.0605, 1.9945, 1.2649, 1.4832, 1.3821, 2.0873, 1.4902, 1.4601, 1.2910, 1.5213, 1.4601, 0.6325, 1.8586, 2.5437, 1.7809, 1.3821, 1.5675, 2.1420, 1.0690, 1.0445, 1.6803, 1.8787, 1.2792, 1.4832, 1.4736, 1.2210, 1.6104, 0.6030, 1.6059, 1.5706, 1.4771, 1.6026, 1.6803, 1.3416, 1.3166, 2.0430, 1.1673, 1.6125, 1.6125, 1.6111, 1.9833, 1.6494, 1.8180, 1.6111, 1.4476, 0.9439, 1.6987, 1.5675, 1.6301, 1.3821, 1.8016, 2.1448, 1.0290, 1.7056, 1.4460, 2.1137, 0.8944, 1.8752, 1.9704, 1.4460, 1.1376, 1.5213, 1.8337, 1.4601, 1.6575, 1.4142, 1.5525, 1.7478, 2.2413, 1.1362, 1.3585, 1.5374, 1.6111, 2.0238, 1.3904, 1.4601, 1.0377, 1.7809, 1.5009, 2.1432, 1.7541, 2.3964, 1.0000, 1.4985, 1.9640, 2.0191, 1.7701, 2.4450, 1.3829, 0.8623, 1.9833, 1.8918, 1.8622, 1.4412, 1.7512, 1.3284, 1.5525, 1.2924, 1.7809, 1.1362, 1.2649, 1.3003, 1.6931, 1.6562, 1.0445, 1.4376, 1.5315, 1.8415, 2.1112, 1.7701, 1.9038, 1.9289, 1.1464, 1.2060, 1.3202, 1.6494, 1.2071, 1.7235, 1.8141, 2.2265, 1.7701, 1.4064, 2.0000, 1.4064, 1.7768, 1.0445, 1.8321, 2.1134, 1.9779, 1.6026, 1.5706, 1.5718, 1.5492, 0.9439, 1.5492, 0.9493, 1.4552, 1.8593, 0.9535, 1.1875, 1.9069, 1.3205, 1.4601, 1.5119, 1.1875, 1.7059, 1.7107, 1.4979, 1.6987, 1.1152, 1.9451, 1.8860, 1.3984, 1.3732, 1.4979, 1.7388, 1.7150, 1.9403, 1.1877, 1.7809, 1.0445, 1.2060, 1.3585, 1.4243, 1.3003, 1.1832, 1.5339, 0.9535, 2.2659, 1.1882, 1.6450, 1.4142, 1.7655, 0.9493, 1.3205, 1.7059, 1.7581, 1.2949, 1.0377, 1.8337, 1.3732, 1.4951, 1.5374, 1.4038, 1.6931, 1.6125, 1.4832, 1.8974, 1.8205, 1.5675, 1.3202, 1.4736, 1.9712, 1.8974, 1.9149, 1.5275, 1.8141, 1.6026, 1.5315, 1.7809, 2.5690, 1.8074, 1.7033, 1.6053, 0.9541, 1.3202, 1.7235, 1.7594, 1.2746, 1.1882, 1.3984, 1.4142, 1.6111, 1.3205, 1.9097, 1.7913, 1.8016, 1.3117, 1.4601, 1.6575, 1.6250, 2.0000, 1.6404, 1.6104, 1.4476, 1.5706, 1.3205, 1.3284, 1.1362, 1.8467, 1.6088, 1.8091, 1.5783, 1.4552, 2.2013, 1.6401, 1.6494, 1.8752, 1.9494, 1.6450, 1.7033, 1.7059, 2.1112, 1.8074, 1.5718, 1.1875, 1.1282, 1.3585, 1.1362, 1.9774, 1.1875, 2.0639, 1.2649, 1.1600, 1.1353, 1.8962, 1.5706, 1.8321, 1.8974, 1.8141, 1.3202, 1.4736, 1.7033, 1.4142, 1.5675, 1.5275, 0.7071, 2.2229, 1.4038, 1.2558, 1.5525, 1.5954, 1.3284, 2.0449, 2.2572, 1.8702, 1.6088, 1.4376, 1.0995, 1.7809, 0.8312, 1.7478, 1.6125, 1.3720, 2.0544, 1.8178, 2.0494, 1.6575, 1.9712, 2.0873, 1.7033, 1.7768, 1.7541, 1.9610, 1.2632, 1.4832, 1.4601, 1.3506, 1.7321, 1.5619, 1.9097, 1.5492, 1.7809, 1.3202, 1.2293, 1.5991, 1.7403, 1.9289, 1.3904, 1.7937, 1.4476, 1.0445, 1.6494, 1.6533, 1.3229, 1.7594, 1.2792, 1.6931, 1.7321, 0.8312, 1.0142, 1.8962, 1.6564, 1.5352, 1.5136, 1.0445, 1.6987, 1.5694, 1.6987, 1.7059, 1.7065, 1.7403, 1.2383, 1.5525, 1.8074, 1.2792, 2.1437, 1.0690, 1.5718, 1.6562, 2.0544, 1.2924, 1.8091, 2.3616, 1.5119, 1.4419, 1.4985, 1.1673, 1.2210, 2.1657, 1.7403, 2.0166, 1.4142, 1.9610, 1.3484, 1.0377, 1.4771, 2.1693, 1.4243, 1.5213, 1.2383, 2.0771, 1.7809, 1.7809, 0.6325, 1.5525, 1.3202, 0.9196, 1.6111, 2.1448, 1.7581, 1.7485, 1.6125, 1.8074, 1.9704, 1.7809, 0.8619, 1.2792, 1.8337, 1.7809, 1.6494, 1.7235, 1.3720, 0.7888, 1.7489, 0.8660, 1.3506, 1.5119 and 1.5213.

From a 6-dimensional Voronoi structure in § 9.6 there are three cells with 2729, 3213, 3246, 3421, 3490, 3606, 3938, 4143, 4398, 4417, 4442 and 4970 vertices. There are two having 1854, 2549, 2557, 2634, 2712, 2720, 2722, 2751, 2804, 2843, 2869, 2878, 2882, 2920, 2931, 2957, 2973, 2996, 3013, 3090, 3260, 3322, 3329, 3343, 3366, 3393, 3418, 3424, 3446, 3513, 3520, 3560, 3577, 3583, 3585, 3610, 3631, 3677, 3714, 3732, 3753, 3767, 3770, 3819, 3835, 3861, 3907, 3919, 3924, 3937, 3939, 4045, 4053, 4091, 4117, 4122, 4123, 4163, 4178, 4219, 4251, 4261, 4269, 4272, 4298, 4317, 4327, 4355, 4461, 4470, 4473, 4474, 4542, 4563, 4572, 4576, 4586, 4618, 4624, 4629, 4640, 4646, 4648, 4700, 4729, 4792, 4810, 4851, 4942, 4975, 4984, 4985, 5058, 5064, 5097, 5161, 5195, 5235, 5287, 5347, 5387, 5455, 5467, 5492, 5529, 5606, 5650, 5838, 5913, 6112, 6193 and 6455 vertices. And there is only one cell with each of the following numbers of vertices, 1198, 1612, 1672, 1686, 1688, 1717, 1727, 1787, 1827, 1864, 1906, 1915, 1921, 1947, 2016, 2052, 2056, 2060, 2123, 2190, 2200, 2223, 2231, 2236, 2267, 2280, 2281, 2286, 2289, 2344, 2353, 2356, 2371, 2372, 2385, 2408, 2423, 2426, 2429, 2455, 2475, 2487, 2499, 2500, 2503, 2504, 2508, 2513, 2542, 2547, 2551, 2560, 2561, 2565, 2575, 2578, 2580, 2585, 2586, 2596, 2617, 2619, 2622, 2650, 2654, 2658, 2664, 2669, 2673, 2686, 2692, 2694, 2704, 2708, 2716, 2718, 2723, 2732, 2733, 2742, 2743, 2755, 2771, 2779, 2781, 2783, 2791, 2800, 2807, 2808, 2811, 2820, 2835, 2837, 2853, 2858, 2864, 2867, 2870, 2873, 2875, 2880, 2884, 2887, 2893, 2894, 2895, 2902, 2907, 2910, 2914, 2916, 2925, 2928, 2934, 2949, 2955, 2956, 2960, 2967, 2974, 2978, 2983, 2985, 2989, 2993, 3007, 3012, 3014, 3027, 3051, 3074, 3102, 3107, 3114, 3116, 3119, 3129, 3130, 3136, 3137, 3141, 3148, 3152, 3164, 3173, 3176, 3180, 3182, 3186, 3188, 3190, 3192, 3199, 3202, 3204, 3206, 3208, 3219, 3221, 3229, 3231, 3237, 3239, 3244, 3245, 3255, 3256, 3259, 3267, 3270, 3271, 3274, 3275, 3285, 3295, 3296, 3305, 3307, 3312, 3316, 3317, 3318, 3319, 3333, 3337, 3344, 3351, 3352, 3354, 3357, 3360, 3370, 3373, 3374, 3378, 3384, 3390, 3394, 3396, 3402, 3403, 3408, 3427, 3428, 3436, 3440, 3443, 3444, 3452, 3453, 3454, 3486, 3487, 3492, 3499, 3502, 3505, 3506, 3509, 3519, 3521, 3522, 3523, 3536, 3541, 3544, 3545, 3547, 3549, 3551, 3555, 3556, 3573, 3575, 3578, 3581, 3582, 3587, 3589, 3594, 3595, 3599, 3604, 3609, 3611, 3620, 3628, 3634, 3635, 3636, 3639, 3647, 3652, 3656, 3658, 3663, 3666, 3668, 3680, 3685, 3686, 3687, 3689, 3691, 3694, 3695, 3696, 3706, 3709, 3710, 3713, 3716, 3717, 3727, 3728, 3735, 3739, 3740, 3742, 3743, 3744, 3751, 3764, 3772, 3775, 3776, 3778, 3782, 3787, 3789, 3792, 3794, 3798, 3803, 3804, 3806, 3808, 3810, 3815, 3818, 3821, 3830, 3834, 3837, 3838, 3842, 3847, 3849, 3850, 3863, 3867, 3868, 3883, 3884, 3889, 3890, 3899, 3900, 3902, 3903, 3908, 3912, 3914, 3918, 3920, 3925, 3929, 3942, 3943, 3944, 3947, 3949, 3956, 3957, 3960, 3962, 3963, 3978, 3980, 3981, 3986, 3988, 3996, 4008, 4016, 4017, 4019, 4027, 4030, 4033, 4035, 4038, 4039, 4044, 4049, 4057, 4072, 4073, 4075, 4077, 4082, 4086, 4092, 4106, 4111, 4112, 4124, 4126, 4128, 4134, 4140, 4145, 4146, 4147, 4148, 4150, 4153, 4157, 4162, 4176, 4177, 4179, 4188, 4189, 4190, 4193, 4195, 4203, 4206, 4212, 4214, 4217, 4218, 4225, 4227, 4232, 4238, 4239, 4244, 4245, 4246, 4247, 4257, 4262, 4271, 4273, 4279, 4283, 4288, 4291, 4293, 4300, 4308, 4313, 4316, 4320, 4322, 4324, 4326, 4331, 4334, 4335, 4337, 4339, 4342, 4346, 4349, 4350, 4351, 4352, 4354, 4357, 4361, 4376, 4377, 4386, 4388, 4395, 4399, 4405, 4410, 4411, 4413, 4425, 4426, 4428, 4447, 4449, 4451, 4453, 4459, 4471, 4475, 4480, 4488, 4493, 4496, 4501, 4504, 4506, 4512, 4515, 4518, 4522, 4523, 4532, 4548, 4552, 4556, 4561, 4574, 4580, 4585, 4587, 4595, 4598, 4602, 4605, 4610, 4614, 4619, 4621, 4626, 4628, 4632, 4634, 4639, 4649, 4651, 4657, 4663, 4665, 4674, 4681, 4684, 4697, 4702, 4703, 4705, 4710, 4711, 4714, 4725, 4728, 4736, 4738, 4740, 4751, 4754, 4755, 4761, 4765, 4770, 4779, 4781, 4814, 4816, 4826, 4829, 4831, 4844, 4850, 4873, 4877, 4880, 4885, 4890, 4892, 4896, 4899, 4900, 4902, 4903, 4905, 4908, 4911, 4914, 4915, 4919, 4924, 4933, 4937, 4944, 4955, 4960, 4967, 4969, 4974, 4992, 4994, 4997, 5000, 5002, 5003, 5007, 5019, 5021, 5028, 5029, 5035, 5039, 5050, 5074, 5080, 5083, 5092, 5093, 5104, 5108, 5110, 5117, 5119, 5120, 5131, 5138, 5156, 5158, 5166, 5167, 5176, 5185, 5192, 5194, 5200, 5203, 5219, 5225, 5226, 5233, 5236, 5239, 5251, 5259, 5265, 5270, 5271, 5274, 5285, 5288, 5290, 5313, 5316, 5318, 5325, 5328, 5336, 5337, 5346, 5364, 5389, 5405, 5408, 5415, 5432, 5439, 5440, 5448, 5457, 5460, 5463, 5464, 5466, 5468, 5470, 5474, 5482, 5484, 5530, 5549, 5567, 5589, 5591, 5598, 5612, 5627, 5632, 5664, 5676, 5680, 5718, 5719, 5720, 5724, 5738, 5744, 5747, 5748, 5749, 5752, 5764, 5802, 5806, 5824, 5841, 5859, 5880, 5885, 5892, 5896, 5899, 5927, 5928, 5947, 5956, 5957, 5966, 5998, 6006, 6014, 6017, 6031, 6035, 6038, 6040, 6054, 6055, 6075, 6081, 6084, 6089, 6092, 6113, 6145, 6166, 6169, 6180, 6192, 6195, 6221, 6237, 6265, 6272, 6273, 6286, 6303, 6305, 6310, 6313, 6320, 6356, 6367, 6456, 6469, 6510, 6515, 6520, 6521, 6524, 6541, 6561, 6609, 6615, 6642, 6677, 6767, 6771, 6831, 6852, 6883, 6918, 6945, 6985, 7056, 7093, 7201, 7379, 7387, 7461, 7560, 7588, 7685, 7764, 7765, 7979, 8308, 8460, 8899, 9079 and 9923.

The following results represent the earlier simulation works. None of the programs used here are used again at the later stage. However there were hundreds of tables similar to the one shown on the following page. These represent the statistical values of the Voronoi structures under study. Here only one page is shown to give an idea of how the project looked at its early stages. The Voronoi structures were then generated by `qhull`, which has been replaced by `voronoin` on Matlab. The latter, though came from the former, is much easier to use.

Result on 700 cells

```
Box size: 10
No compression
Number of cells: 700
Number of vertices: 4380
Number of cells in frame: 341
Time for counting stats: 2191.39 seconds
Number of faces connected to the first vertice at infinity: 181
Time for finding cell volumes: 3.1500 seconds
```

The table which follows was drawn by TEX's `plain` macro written by Donald Knuth who is the creator of TEX. This thesis is compiled with the help of Knuth's `manmac` macro and the *thshead* macro that I have written. Both the `tdhead` that I have used for typesetting my books (Tiyapan, 2003, KNT8(iii) to KNT8(ix)). My book Voronoi Translated (Tiyapan, 2001, KNT8(ii)), however, used only `manmac`. I am now using `bkhead`, which has been developed from `tdhead`. I am now writing a book on TEX in which you may be able to find `bkhead` and `tdhead` being discussed in more details.

n_c 700	n_v 4,380	$n_{f,1^{st}v}$ 181	$n_{c_{in}}$ 341	$t_{CPU,stat.}$ 2191.39 sec.	$t_{CPU,A}$ 137.13 sec.	$t_{CPU,V}$ 3.15 sec.	
$\min n_{v_c}$ 10	$\max n_{v_c}$ 46	$n\bar{v}_c$ 25.106	$\sigma^2_{n_{v_c}}$ 46.784	$\sigma_{n_{v_c}}$ 6.8399	$g, \bar{n}_{v_c}$ 24.169	$h, \bar{n}_{v_c}$ 23.208	n_{v_c}
$\widetilde{n_{v_c}}$ 24	$\delta_\mu(n_{v_c})$ 5.4023	$M^2(n_{v_c})$ 46.717	$M^3(n_{v_c})$ 1.4240×10^2	$M^4(n_{v_c})$ 6.5877×10^3	$\kappa_{n_v,c}$ 3.0184		
$\min n_{v,c_{in}}$ 10	$\max n_{v,c_{in}}$ 46	$v\bar{c}_{in}$ 26.246	$\sigma^2_{v,c_{in}}$ 40.974	$\sigma_{v,c_{in}}$ 6.4011	$\overline{V}_{,gc_{in}}$ 25.464	$\overline{V}_{,hc_{in}}$ 24.668	$n_{v,c_{in}}$
$\widetilde{n}_{v,c_{in}}$ 26	$\delta_\mu(n_{v,c_{in}})$ 5.1101	$M^2(n_{v,c_{in}})$ 40.854	$M^3(n_{v,c_{in}})$ 1.0338×10^2	$M^4(n_{v,c_{in}})$ 4.9173×10^3	$\kappa_{n_{v,c_{in}}}$ 2.9461		
$\min n_{\aleph_v,c_{in}}$ 8	$\max n_{\aleph_v,c_{in}}$ 26	$\bar{n}_{\aleph_v,c_{in}}$ 16.123	$\sigma^2_{n_{\aleph_v,c_{in}}}$ 10.244	$\sigma_{n_{\aleph_v,c_{in}}}$ 3.2006	$\bar{n}_{g,\aleph_v,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_v,c_{in}}$ 15.495	$n_{\aleph_v,c_{in}}$
$\widetilde{n}_{\aleph_v,c_{in}}$ 16	$\delta_\mu(n_{\aleph_v,c_{in}})$ 2.5550	$M^2(n_{\aleph_v,c_{in}})$ 10.214	$M^3(n_{\aleph_v,c_{in}})$ 12.922	$M^4(n_{\aleph_v,c_{in}})$ 3.0733×10^2	$\kappa_{n_{\aleph_v,c_{in}}}$ 2.9461		
$\min n_{\aleph_e,c}$ 8	$\max n_{\aleph_e,c}$ 27	$\bar{n}_{\aleph_e,c}$ 15.669	$\sigma^2_{n,\aleph_e,c}$ 12.439	$\sigma_{n,\aleph_e,c}$ 3.5269	$\bar{n}_{g,\aleph_e,c}$ 15.279	$\bar{n}_{h,\aleph_e,c}$ 14.894	$n_{\aleph_e,c}$
$\widetilde{n}_{\aleph_e,c}$ 15	$\delta_\mu(n_{\aleph_e,c})$ 2.7792	$M^2(n_{\aleph_e,c})$ 12.422	$M^3(n_{\aleph_e,c})$ 22.761	$M^4(n_{\aleph_e,c})$ 4.8871×10^2	$\kappa_{n,\aleph_e,c}$ 3.1673		
$\min n_{\aleph_e,c_{in}}$ 8	$\max n_{\aleph_e,c_{in}}$ 26	$\bar{n}_{\aleph_e,c_{in}}$ 16.123	$\sigma^2_{n(\aleph_e),c_{in}}$ 10.244	$\sigma_{n(\aleph_e),c_{in}}$ 3.2006	$\bar{n}_{g,\aleph_e,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_e,c_{in}}$ 15.495	$n_{\aleph_e,c_{in}}$
$\widetilde{n}_{\aleph_e,c_{in}}$ 16	$\delta_\mu(n_{\aleph_e,c_{in}})$ 2.5550	$M^2(n_{\aleph_e,c_{in}})$ 10.214	$M^3(n_{\aleph_e,c_{in}})$ 12.922	$M^4(n_{\aleph_e,c_{in}})$ 3.0733×10^2	$\kappa_{n(\aleph_e),c_{in}}$ 2.9461		
$\min n_{\aleph_f,c}$ 8	$\max n_{\aleph_f,c}$ 27	$\bar{n}_{\aleph_f,c}$ 15.629	$\sigma^2_{n(\aleph_f),c}$ 12.274	$\sigma_{n(\aleph_f),c}$ 3.5034	$\bar{n}_{g,\aleph_f,c}$ 15.242	$\bar{n}_{h,\aleph_f,c}$ 14.857	$n_{\aleph_f,c}$
$\widetilde{n}_{\aleph_f,c}$ 15	$\delta_\mu(n_{\aleph_f,c})$ 2.7635	$M^2(n_{\aleph_f,c})$ 12.256	$M^3(n_{\aleph_f,c})$ 21.051	$M^4(n_{\aleph_f,c})$ 4.6791×10^2	$\kappa_{n(\aleph_f),c}$ 3.1149		
$\min n_{\aleph_f,c_{in}}$ 8	$\max n_{\aleph_f,c_{in}}$ 26	$\bar{n}_{\aleph_f,c_{in}}$ 16.123	$\sigma^2_{n(\aleph_f),c_{in}}$ 10.244	$\sigma_{n(\aleph_f),c_{in}}$ 3.2006	$\bar{n}_{g,\aleph_f,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_f,c_{in}}$ 15.495	$n_{\aleph_f,c_{in}}$
$\widetilde{n}_{\aleph_f,c_{in}}$ 16	$\delta_\mu(n_{\aleph_f,c_{in}})$ 2.5550	$M^2(n_{\aleph_f,c_{in}})$ 10.214	$M^3(n_{\aleph_f,c_{in}})$ 12.922	$M^4(n_{\aleph_f,c_{in}})$ 3.0733×10^2	$\kappa_{n(\aleph_f),c_{in}}$ 2.9461		
$\min A_c$ 0.59209	$\max A_c$ 83.165	$\bar{A}_c$ 3.4086	$\sigma^2_{A_c}$ 2.0373	σ_{A_c} 2.3046×10^2	$\bar{A}g, c$ 5.0726	$\bar{A}h, c$ 1.0502	A_c
$\widetilde{A}_c$ 1.1935×10^2	$\delta_\mu(A_c)$ 4.0384	$M^2(A_c)$ 2.8300	$M^3(A_c)$ 2.0530	$M^4(A_c)$ 1.3800	κ_{A_c} 2.0572		
$\min A_{c_{in}}$ 0.59209	$\max A_{c_{in}}$ 3.4086	$\bar{A}_{c_{in}}$ 2.0373	$\sigma^2_{A_{c_{in}}}$ 1.0502	$\sigma_{A_{c_{in}}}$ 4.0384	$\bar{A}g, c_{in}$ 2.8300	$\bar{A}h, c_{in}$ 2.0530	$A_{c_{in}}$
$\widetilde{A}_{c_{in}}$ 1.3800	$\delta_\mu(A_{c_{in}})$ 2.0572	$M^2(A_{c_{in}})$ 2.9782	$M^3(A_{c_{in}})$ 3.9125	$M^4(A_{c_{in}})$ 3.1031	$\kappa_{A_{c_{in}}}$ 1.0800		
$\min A_{fr,c}$ 8.5015×10^{-8}	$\max A_{fr,c}$ 1.1941×10^{-5}	$\bar{A}_{fr,c}$ 4.8942×10^{-7}	$\sigma^2_{A_{fr,c}}$ 2.9253×10^{-7}	$\sigma_{A_{fr,c}}$ 3.3091×10^{-5}	$\bar{A}g, fr, c$ 7.2835×10^{-7}	$\bar{A}h, fr, c$ 1.5079×10^{-7}	$A_{fr,c}$
$\widetilde{A}_{fr,c}$ 1.7137×10^{-5}	$\delta_\mu(A_{fr,c})$ 5.7986×10^{-7}	$M^2(A_{fr,c})$ 4.0635×10^{-7}	$M^3(A_{fr,c})$ 2.9478×10^{-7}	$M^4(A_{fr,c})$ 1.9816×10^{-7}	$\kappa_{A_{fr,c}}$ 2.9538×10^{-7}		
$\min A_{fr,c_{in}}$ 7.5700×10^{-4}	$\max A_{fr,c_{in}}$ 4.3579×10^{-3}	$\bar{A}_{fr,c_{in}}$ 2.6048×10^{-3}	$\sigma^2_{A_{fr,c_{in}}}$ 1.3427×10^{-3}	$\sigma_{A_{fr,c_{in}}}$ 5.1632×10^{-3}	$\bar{A}g, fr, c_{in}$ 3.6182×10^{-3}	$\bar{A}h, fr, c_{in}$ 2.6248×10^{-3}	$A_{fr,c_{in}}$
$\widetilde{A}_{fr,c_{in}}$ 1.7644×10^{-3}	$\delta_\mu(A_{fr,c_{in}})$ 2.6301×10^{-3}	$M^2(A_{fr,c_{in}})$ 3.8077×10^{-3}	$M^3(A_{fr,c_{in}})$ 5.0023×10^{-3}	$M^4(A_{fr,c_{in}})$ 3.9674×10^{-3}	$\kappa_{A_{fr,c_{in}}}$ 1.3808×10^{-3}		
$\min V_c$ 0.88511	$\max V_c$ 6.2052×10^6	$\overline{V}_c$ 2.7455×10^4	$\sigma^2_{V_c}$ 8.9303×10^{10}	σ_{V_c} 2.9884×10^5	$\overline{V}g, c$ 22.548	$\overline{V}h, c$ 6.3296	V_c
$\widetilde{V}_c$ 7.6212	$\delta_\mu(V_c)$ 5.1126×10^4	$M^2(V_c)$ 8.9176×10^{10}	$M^3(V_c)$ 4.3202×10^{17}	$M^4(V_c)$ 2.3781×10^{24}	κ_{V_c} 2.9904×10^2		
$\min V_{c_{in}}$ 0.88511	$\max V_{c_{in}}$ 1.8958×10^3	$\overline{V}_{c_{in}}$ 20.542	$\sigma^2_{V_{c_{in}}}$ 1.5264×10^4	$\sigma_{V_{c_{in}}}$ 1.2355×10^2	$\overline{V}g, c_{in}$ 5.9195	$\overline{V}h, c_{in}$ 4.4392	$V_{c_{in}}$
$\widetilde{V}_{c_{in}}$ 5.4355	$\delta_\mu(V_{c_{in}})$ 27.858	$M^2(V_{c_{in}})$ 1.5219×10^4	$M^3(V_{c_{in}})$ 2.2911×10^7	$M^4(V_{c_{in}})$ 3.9178×10^{10}	$\kappa_{V_{c_{in}}}$ 1.6915×10^2		
$\min V_{fr,c}$ 4.6056×10^{-8}	$\max V_{fr,c}$ 0.32288	$\overline{V}_{fr,c}$ 1.4286×10^{-3}	$\sigma^2_{V_{fr,c}}$ 2.4179×10^{-4}	$\sigma_{V_{fr,c}}$ 1.5550×10^{-2}	$\overline{V}g, fr, c$ 1.1733×10^{-6}	$\overline{V}h, fr, c$ 3.2935×10^{-7}	$V_{fr,c}$
$\widetilde{V}_{fr,c}$ 3.9656×10^{-7}	$\delta_\mu(V_{fr,c})$ 2.6603×10^{-3}	$M^2(V_{fr,c})$ 2.4145×10^{-4}	$M^3(V_{fr,c})$ 6.0865×10^{-5}	$M^4(V_{fr,c})$ 1.7433×10^{-5}	$\kappa_{V_{fr,c}}$ 2.9904×10^2		
$\min V_{fr,c_{in}}$ 1.2636×10^{-4}	$\max V_{fr,c_{in}}$ 0.27064	$\overline{V}_{fr,c_{in}}$ 2.9326×10^{-3}	$\sigma^2_{V_{fr,c_{in}}}$ 3.1108×10^{-4}	$\sigma_{V_{fr,c_{in}}}$ 1.7638×10^{-2}	$\overline{V}g, fr, c_{in}$ 8.4507×10^{-4}	$\overline{V}h, fr, c_{in}$ 6.3374×10^{-4}	$V_{fr,c_{in}}$
$\widetilde{V}_{fr,c_{in}}$ 7.7598×10^{-4}	$\delta_\mu(V_{fr,c_{in}})$ 3.9770×10^{-3}	$M^2(V_{fr,c_{in}})$ 3.1017×10^{-4}	$M^3(V_{fr,c_{in}})$ 6.6659×10^{-5}	$M^4(V_{fr,c_{in}})$ 1.6273×10^{-5}	$\kappa_{V_{fr,c_{in}}}$ 1.6915×10^2		

C

§ C. Publications and submissions of papers

§ C.1 Critical probability of 2-d tessellation

Critical probability and other properties of 2-d tessellation†

Kittisak Tiyapan‡ G. A. Davies§ David J. Bell‡

2^{nd} September 1996

Abstract

The critical probabilities of Voronoi tessellation and some other uniform lattices, including triangular, square, honey-comb, and Kagomé lattices, are identified by mean of an algorithm developed. The properties of these uniform lattices agree with existing well known results. This work provides the bond critical probability for Voronoi tessellation.

Introduction

A tessellation is an aggregate of cells that cover the space without overlapping. A Voronoi polygon is also known as a Dirichlet polygon, a Wigner-Seitz polygon, a Theissen polygon, a Blum's transform, an S polygon, a cell model, a plant polygon, Wirkkungsbereich, etc.

Definition 1. *Let* Φ *be a distribution of a countable set of nuclei* $\{x_i\}$ *in* R^d*, and let* $x_1, x_2, x_3, \ldots$ *be the coordinates of the nuclei. Then, the Voronoi region is*

$$\Pi_i = \{x | d(x, x_i) < d(x, x_j) \forall j \neq i\}$$

where $d(x, y)$ *is the Euclidean distance between* x *and* y*.*

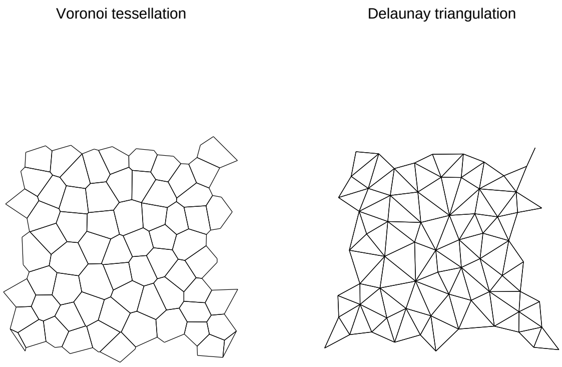

Figure 1.. Duality of Voronoi tessellation and delaunay triangles.

The number of Voronoi points (nuclei) and edges are both $\mathcal{O}(n)$ where $n \in N$, and the number of Delaunay triangles and edges are also $\mathcal{O}(n)$ (Ahuja and Schachter, 1983).

A bond-problem of any lattice L can be translated into a site-problem on L^c, its covering lattice constructed (Shante and Kirkpatrick, 1971) by the following procedure,

1 . Replacing each β_i by α_i^c.

2 . Creating β_i^c by the rule that $\forall \alpha_i^c, \alpha_j^c \in L^c$, α_i^c and α_j^c are connected by a bond of L^c if and only if their corresponding β_i and $\beta_j \in L$ have a common terminal atom of L.

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† This pape was presented at the MTNS96 conference held in St. Louis, U.S., June 1996, by the first author.

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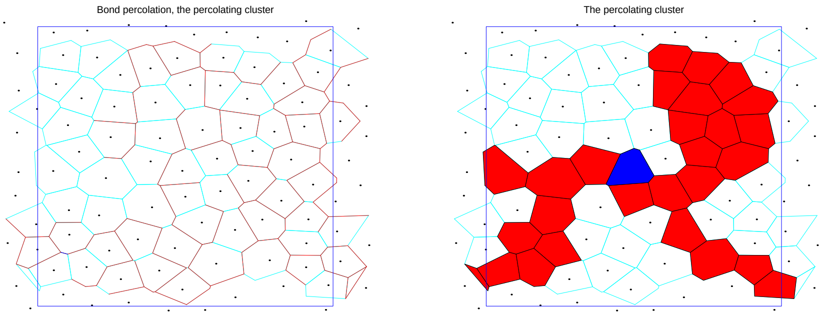


(a) *Example of a bond percolation* (b) *Example of a site percolation*

Figure 2 *Example of* (a) *a bond and* (b) *a site percolation.*

A list of applications can be summarised into two groups according to the two main groups. *Bond percolation* finds application in hydrology (movement of water in dam structure, intrusion of sea water in coastal areas, filter beds, *etc*), petroleum engineering (petroleum and natural gas production, exploration, logging, *etc*), chemical engineering (heterogeneous catalysis, flow through packed beds, gel permeation chromatography, porous polymer films used in separation processes, biological membranes, inorganic membranes, *etc*), medicine and biomedical engineering (biological membranes, biological filters, flow of blood and other body fluids, electro-osmosis, *etc*), electrochemical engineering (porous electrodes, permeable and semipermeable diaphragms for electrolytic cells, *etc*), and in communication (performance of communication networks with blockage). *Site percolation* finds applications in permeation through filtration membranes (Bell *et al*, 1995), sieve blinding (Wilkinson and Davies, 1989), membrane fouling, (for example, of the Anotec (ANOPORE) microfiltration membrane) and in the form of pore clogging, effect of back-flushing and crossflow microfiltration have been studied and close agreement with experiments was obtained.

At and above the *critical probability* p_c (Shante and Kirkpatrick, 1971) a percolating cluster, a cluster that spans infinite length, occurs. A mathematical definition of p_c can be found in Bousquet-mélou (1996). The exact value of p_c for some of the uniform lattices can be found by the method of series expansion (Onody and Neves, 1992; De'Bell and Essam, 1983). But for a Voronoi lattice, at this moment, there is no deterministic method. The only possible way is by doing simulations on the lattice as has been done here.

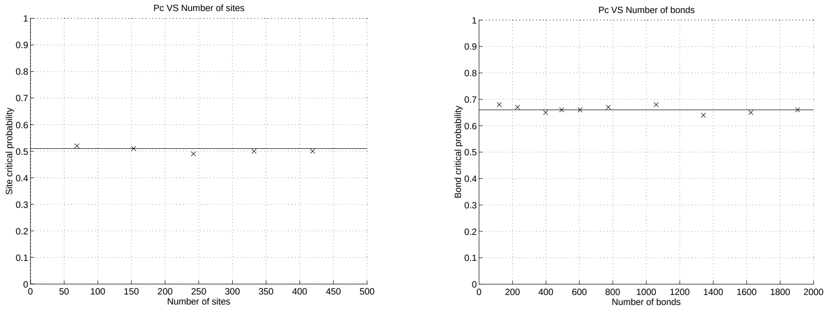
Results

Simulations are made by a developed algorithm (Tiyapan, 1995) based on Monte Carlo method. The results are best presented graphically.

Here $p^s_{c,\mathrm{avg}}$ and $p^b_{c,\mathrm{avg}}$ are the critical probability of site, and bond percolation respectively, averaged over a reasonably large amount of simulation; $n(i)$ is the number of clusters which have i members, where $i \in I$; n is the total number of elements, that is sites for a site problem and bonds for a bond problem; $n(i)_{\max}$ is the size of the largest cluster at a specified p; and p is the ratio between the number of blocked elements and the total number of elements. The coordination numbers for a Voronoi lattice is assumed to be the number of neighbours of each element, averaged over every element within the network. To make clearer a pictorial demonstration of the result, in most of the pictures only the first cluster which percolates is shown.

Discussion

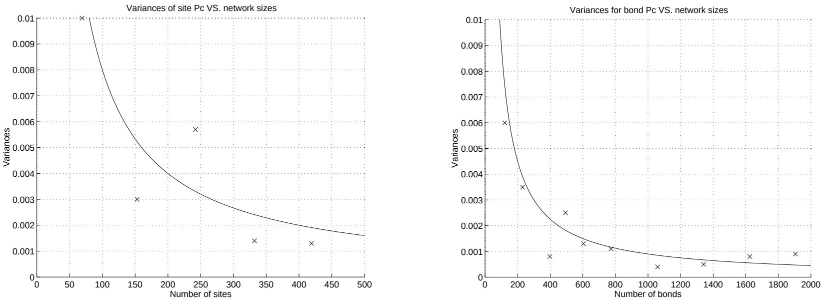
From Figure 3 (a) and (b) it is clear that the critical probability is independent of network size. Thus p_c is a property that is intrinsic for each type of network, and further studies show that it differs from one type of network to another. The idea of infinite cluster is also confirmed.



(a) (b)

Figure 3 (a) $p^s_{c,avg}$*'s plotted against number of sites, (b)* $p^b_{c,avg}$*'s plotted against number of bonds*

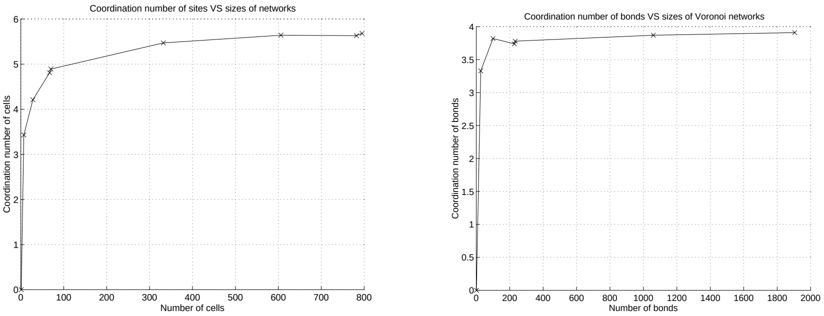
Figure 4 (a) and (b) show that the value of p_c's can be accurately obtained by either doing one simulation on a very large network, or by doing many simulations on a smaller network, since they show that the variance of the results reduces with increasing sizes.



(a) (b)

Figure 4 (a) $\sigma^2_{p^s_c}$*'s, and* (b) $\sigma^2_{p^b_c}$*'s, plotted against sizes of networks.*

Coordination numbers (mean contact numbers) of sites and bonds of Voronoi networks are plotted against network sizes in Figure 5 (a) and (b) respectively. The values seem to be approaching, but never reaching, 6 and 4 for site and bond problems respectively.



(a) (b)

Figure 5 (a) *Coordination number of sites, and* (b) *Coordination number of bonds, plotted against network sizes.*

Figure 6 shows summary of p_c's obtained from all simulations.

Figure 6 [The files for these pictures are lost.] p_c*'s obtained for (a) square lattice,* 220 *bonds,* $p^b_c = 0.53$, $p^b_{c,avg} = 0.47$*; (b) triangular lattice,* 305 *bonds,* $p^b_c = 0.33$, $p^b_{c,avg} = 0.34$*; (c) honeycomb lattice,* 111 *bonds,* $p^b_c = 0.65$, $p^b_{c,avg} = 0.64$*; (d) Kagomé lattice,* 174 *bonds,* $p^b_c = 0.55$, $p^b_{c,avg} = 0.52$*; (e) [399 nuclei, 240 cells,* $p_c = 0.47$*, limit* $= 0.04$*] site problem for Voronoi lattice,* $p^s_{c,avg} = 0.51$*; and (f) [400 nuclei, 324 cells, 1044 bonds, bond percolation, limit*$= 0.04$, $p_c = 0.65$*] bond problem for Voronoi lattice,* $p^b_{c,avg} = 0.66$.

Conclusion

Critical probability is a value which is intrinsic to each type of networks. The value for each uniform or random lattices is constant and does not depend on the size of network.

[All the simulation] result shown in this paper was done by MATLAB running on UNIX workstations. The program used for the generation of Voronoi lattices was adapted from Jafferali (1995). A Kagomé lattice can be generated from either a triangular lattice or a honey-comb lattice (Tiyapan, 1995).

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§ C.2 Abstracts of books I wrote

Interesting English. ISBN 974–346–182–5

This book is written in Thai with the title *Bhạṣạ angkṛiṣ an nạ̀soncaị* during the time of economic gloom for both myself and my country Thailand. It is built around three major experiences which I consider to be the turning points of my life, namely my studying the Sixth Form English in New Zealand (1983), my acquaintance with a British scholar Amnad Khitapanna since 1985, and my experience in Budapest and Europe (1991). In 104 pages I try to pass on as much as many things as possible those things which I consider valuable from all the three sources mentioned. With my teaching a class in English Literature at the Pradíṣṭh Center in Bangkok as a catalyst the book was published at the Chulalongkorn University Press (2000) under the trademark *Kittix* which has now become my own. From my study of the Sixth Form Engish comes the sections on false friends, alliteration and assonance, onomatopoeia, jargons, Macbeth, several poetic quotations, and the section on books and movies, from discussions with Amnad the seemingly prefixed words for example *dishevelled* and *unkempt*, and songs by Andrew Lloyd Webber, and from my experience of working in Eastern Europe during 1991 the game *twenty questions*. The rest of the materials are mainly what I used in teaching the English class mentioned. They are tragedies *vs* comedies, Titus Andronicus, Julius Caesar, Romeo and Juliet, poems by Don Marquis, and homophones. Apart from that this book also mentions some word games the examples of which are riddles, aeiou-ordered words (for example, *abstemious*), spoonerisms, tongue-twisters, anagrams, scrabbles, crosswords, Targets, doublets, syzygies, word squares, and acrostics. There is a figure which shows the relative degrees of adjectives on a straight line, and also a Venn diagram showing the comparative domains of the English words *wok* and *pan*, and the Thai word *kada*. I had no laser printers and camera-ready copy of the book was printed from a laser printer at the Control System Engineering group at the Electrical Engineering Department, Chulalongkorn University with the courtesy of my former supervisor Assistant Professor Watharapong Khovidhungij [Wacȧ̀rȧ̀bongṣ́ Ḳhọviḍụrkic] whom I deeply thank. But the quality of the printing is not very good, which results in some parts of the book being difficult to read. To answer the question I have been asked, the picture in the biography section was from the plastic badge which I used to wear when working at Jasmine in Bangkok.

Free translation of English. ISBN 974–346–765–3

The title of the book written above is the English translation of the actual title *Plạe kled́ Angkṛiṣ*, the word *plạe kled́* being an adaptation from the jargon of the old thai sword fighting *f̣an kled́* which means *an improvising practice*, or to do this when the word is a verb. Again the camera-ready copy being printed at the Control System Engineering Department except this time with the courtesy of Dr. Manop Wongsaisuwan who is my senior both at the Electrical Engineering Department, Chulalongkorn Unversity, as well as at the Furuta Laboratory, Tokyo Institute of Technology, whom I consider a wizard in the computational matters, and to whom I thank. This is a book about translation written from my experience in translating and inspired by my teaching in a translating class at the same Pradíṣḍh Center already mentioned. It is a book of translation both from Thai into English as well as from English into Thai. As such, none of the various examples given is easy and straight forward, and all of them have baffled some translators in the past. It was written at the height of my artistic and literarical ability and at the time when I also enrolled in and attended classes in European (English included) literatures at the Ramkamhaeng University in Bangkok. The book starts off with the phrase *kạrplạe gụe kạrplạe gụe kạrplạe*, literally *translation is translation is translation* but which I deliberately translated as *to write is to write is to write* all languages and writings being but translations of the thought. Historically the romans seldom translated but used Greek in their writings instead, and the greeks never did for theirs were the originals. The book talks about creativity in translation and plagiarism. In general rules may be broken at the right time. Styles of writing are mentioned again without repetition of my previous book. Namely these are the registers and modes of use, alliterations, assonances, onomatopoeias, metaphor, simile, repetition, coupling, rhetoric, allusions, direct and indirect speeches, mythological materials, burlesques, comparative usages of punctuation marks of both languages. The books says that simplicity is the mother of invention. Archaism and parallellism are mentioned, and so are jargons from various disciplines. Examples from literature includes those by Shakespeare, Lewis Caroll, Yeats, and Sundaurbhụ̀. In particular the inventive usages of Cockney English is problematic in translation, no negative pronouns exists in Thai so one needs to try hard to find their equivalences. Some of the jargons I mentioned are those of thai arts, fruits, spices, illnesses, anatomy, fauna, and flora. Some of the humourous clippings from the internet obtained from my friend and computer professional Ken Labinjok in Bristol are also shared with the readers in Thai, some of which needed some explanation to become funny. As for the rest, most of the things I wish to say the book has hardly left out; I would never have thought that I had put so many of them in these 106 pages had I not looked at it again just now. By the way, does anyone happen to know that *novemdecillion* in American is 10^{60} but in English is 10^{114} I wonder. Both my first book and my second one have one problem alike, that is that the computer that I use is of japanese make and can fluently write Japanese but not Thai. One problem which persisted and baffled me throughout is the mysterious disappearance of all the *ỵ*'s. Ironically enough the alphabet *ỵ* is in Thai called *ỵạu phụ̀ỵing*, literally *w for women*! The page on biography is a little more comprehensive than that of the first book. Having mentioned a picture in the previous one, the picture in this book was taken in front of my house in

Bangkok.

Voronoi Translated. (ISBN 974–13–1503–1)

That succinct title was followed by a more descriptive one, *Introduction to Voronoi tessellation and essays by G. L. Dirichlet and G. F. Voronoi*, which summarises this book. This is the first book, and hopefully the last one, I have written which has a grey cover, *grey* being the symbol of *Gandalf the Great* which in turn rhymes with *Graham Davies* the name of my supervisor and sponsor from end 2000. It is also the first book which I typeset with TeX which I consider complicated but works, while both of the previous two books were typeset using LaTeX which I consider simple and works, the reason for the change being the upgrade of the latter to LaTeX2ε which I consider complicated and (presumeably) works together with the fact that it has TeX working in the background. The switching over thus helps reduce one shell around the equivalences of the only desirable attribute left which is that of *complicated but works.* Here the two venerable alphabets *v* and *n* rules throughout, starting from the title *Voronoi*, the name of a russian mathematician of the nineteenth century, then the picture taken in *Vienna* on the front, and that in *Venice* on the back covers; if you look carefully enough, somewhere in the book you will find another picture of a stone masonry with a caption saying *Verona* which Professor Davies decidedly says is not a Voronoi tessellation. The introduction gives some historical backgrounds and pictures of various orders of covering lattices of the Voronoi tessellation in two dimensions, which can be used to represent for example a realistic picture of the conglomeration of grains within some kind of matrix. The triangular and the hexagonal lattices are examples of dual lattices. The picture of the hexagonal and kagome lattices superimposed on top of each other is not a picture of the Eden Project in Cornwall but an example of covering lattices. This is not a free translation of the seminal works by G. F. Voronoï and G. L. Dirichlet. I tried to retain the original as much as possible, namely where I thought the grammar in the original was incorrect or the structure of the sentence is convoluted I coined up the equivalence, though this is by no mean how I always translate. As a result the translation is necessarily more difficult to read than it would have been had it been freely summarised. My reason for doing it in this style is firstly in order to transfer as much nuances across as possible, and secondly simply because I am not a mathematician and therefore in no position to summarise the work of one. Last but not least must be my own shortcoming and inexperience in translating from French and German. The preface of the book gives a brief history of the *Journal für die reine und angewandte Mathematik* while the introduction that of the Voronoi tessellation. Also one knows from the latter that the name *kagome* is a Japanese word which means *basket interstice or pattern*, the fact that I realised while reading the Japanese language in Tokyo. *Me* is the word for *eye* as well as *pattern*, and in Japnanese *kago* means *an intersticed basket* as much as *taklạ̀* does in Thai. There is hardly discipline in which Voronoï tessellation has not found applications and the list of relevant areas is endless, ranging from management to computation to physics. This book has 282 pages of contents.

Percolation within percolation and Voronoi Tessellation. ISBN 974–91036–1–0

This book contains both my original thesis, as submitted to UMIST in March 2003, as well as the translation of Schumann's Liederkreis, Op. 24, from German into English and Thai that I did for Aṃnạc Gịtȧbarrṇạ who was once the president of the Bangkok Music Society. The preface says that this is not the final version of the thesis. In fact only God knows what the final thesis will look like. There are 585 pages in all, and approximately half of these are appendices. As usual I write this book and thesis in TeX. As such it is my most complex project on TeX to date. A TeX macro I wrote is given which in its packed form is over 600 lines long. It predominantly deals with how to display pictures, as well as index and cross references. Another TeX macro is also given which contains my works on transcription systems of various languages from Lanna to Chinese to ASL, that is the American Sign Language. Other programs given are what I had used on Matlab to produce the results mentioned in the contents. The introduction and literature survey are eclectic verging on desultory, which is partly due to the multidisciplinary nature of the work. Parts of the introduction contain the contents of the emails I wrote to various people concerned when there were serious problems affecting the progress of the work. Future researchers and academicians may be able to learn from this and avoid facing or incurring them. The literature survey is distributed throughout the book instead of centralisedly staying together in one place. I give some flavour of the different disciplines concerned. These include mathematics and geometry, physics and percolation, statistics and statistical physics, tessellation and Voronoi Tessellation, quadratic forms and quadratic equations. Observing the clusters and their development is important in the study of percolation. The sudden accelerated growth of the cluster size heralds the onset of percolation. Applications of percolation theory to traffic congestion and economics transition are also mentioned. The value of percolative thresholds are obtained from the simulation for the Voronoi Tessellation, 2 homohedral tilings, cities traffic, and *n*-gons and spheres in continuum. Also in the appendices are the collection of my writings both technical and nontechnical since 1994. The range of the topics of these is rather wide, for example antimony trioxide extraction, cyberspace, computer worm, economic and traffic modellings, variable structure control and singular perturbation. There are also articles on languages. And I have included translations I have made of the seminal works by Dirichlet and Voronoi on the Voronoi Tessellation.

Thai grammar, poetry and dictionary. ISBN 974–17–1861–6

In this book I present my a new transcription system for Thai using the roman alphabet. The tonal symbols used in this system are adopted from the *pīnyīn* system of Chinese. The consonants either follow the conventions in Pali and Sanskrit or are invented by means used in these two languages. These latter ones are namely /[ḳh, z, y̦, ḍ́, d́, ḅ, fh, f, ‘.] and *ḥ*. Not only all the consonants but also all the romanised vowels are standardised such that they represent an isomorphism, that is one-to-one and onto, of the Thai script. The usefulness of this is immense because one can then write in English and yet refer to any Thai words no matter how difficult with no ambiguities and with ease. In order to prove the usability of this system, I have used it to write no less than 100 examples of all the kinds of poem that I know. Also the dictionary at the end of the book in a way provides another proof of its user-friendliness. For many reasons some of which are beyond my control while others could and should have been avoided had I been more experienced, there are *many* spelling errors in the book. Some of these are the following, each correct word being given following the page number where it appears, and in brackets its meaning and misspelt form: 13, Rạmgaṃhạeng ([name of a king], Rạmgamhạeng), 15, khoad́ (bottle; khoadt); 18, pàṃm̈pọ́eë (lapse-some; pàṃm̈pọ́ee), dharrm (dhamma; damr), dharrmȧd́ạ (normal; damrȧd́ạ); 19, ạu ([a long vowel]; ạ̇u [a hidden vowel]) (this error, since it is in the macro, occurs everywhere, therefore there are two possibilities if you see an *ạ̇u* in the book and most of the times it means this *ạu*); 22, ywạk (to lust; yạk), d́uan (moon; d́oen); 24, lāo (a pen for animals; lạ̄); 25, tàung (have to; tạ̀ng), dondạn (enduring; tontạn), saṃbhao (a Chinese junk; saṃghao); 26, aukmạ (coming out from; aumạ); 27, graị (who; gaị); 29, d́ị (good; dị), tọayặng (example; tụayặng); 43, c_{1l} ([defined to be one of the 24 low leading consonants]; $c = 1l$); 45, $\circ_1$ ([the first syllable to be rhymed with in this stanza]; $\circ, 1$), ḅặu (not; bặu); 46, ḅauk (tell; bauk); 47, ngạn (work, hgạn), glạ̄ew (miss by a hair breadth; gạ̄ew); 48, graị (anybody; grai); 51, Ịk (and also; Ḷk), Ḅanggom ([homage done while sitting on one's own knees]; Ḅanggkom); 55, brāum (together with; grāum); 57, jọ̀ay (to help; jọ̀y); 58, prayọjṇa (usefulness; prayọjṇa), lavēn (avoid; lavẹ̀n); 62, tặe (but; tặen); 64, d́ek (child; dek); 65, Sạmon ([a literarical character]; Saq.mon); 69, kayjạ (marijuana; kayyjạ); 71, mȧhọ‘.ạr (enormous; mȧhọlạr), sȧnạmkị‘.ạ (sport stadium; sȧnạmkịlạ), sȧmạgom (society, club; lȧmạgom); 72, y̦hặị (big; y̦hāi); 74, jāo (morning; jạ̄); 75, lòang (breach; lọ̀ng); 77, thoed́ (Please!; toedz.); 80, śọkạ (sadness; śọkjạ); 81, krạḅvhài (pay homage to; klạḅvhài); 84, Bhȧgȧvadȧgịtạ ([the Sanskrit epic *Bhagavad Gītā*]; Bhȧgȧvatȧgịtạ); 87, muddạ (print; mudadạ); 88, durẹś (feel disgusted; dhurẹś); 90, bloen (enjoying, absorbed; boen); 97, ywạk (want; yhạk); 100, ḅặu (not; ḅặu). The aleph symbols are for example, in pages 41, 5.19; in page 42 respectively 5.1 and 5.2. As quoted at the end of the book, ‘Es gibt Dinge, die findet man nur, wenn man etwas ganz anderes gesucht hat’, may be you will find something good from the book after you have searched for all the errors it contains! I shall be happy if you let me know this is the case.

A Lanna in town. ISBN 974–17–1860–8

This is the first one of the pentalogy I wrote during the first half of 2003. The others are *A Kiwi Lanna, A British Lanna, Edokko no Lanna* and *The Siamese Lanna.* All these five books have exactly the same length, that is 112 pages two of which contains no contents. A simple process of addition tells us that they would have formed a 550-page volume if put together into one book. I want to do this in the future when I shall correct all the errors in them, spelling and all. These books contain only two themes one of which is the things I relate from my experiences while the other what they teaches me and what research I have done in order to understand them. The part which provides or tries to give understandings is definitely a more interesting one. But it is difficult to put it into the abstract without going into such a great length as including the whole book. The setting of this book begins in Thailand where I was born. Then it goes on to New Zealand, Bangkok, Surrey, Budapest, Manchester, and Tokyo. These are only a nonredundant listing of those places where I have lived. It is already too complicated without my trying to put all the other places to where I have visited.

A Kiwi Lanna. ISBN 974–91237–3–5

I was lucky to have become a part of the *saṃnakd́ạḅ* (sword school) Śrị Ȧyudhyạ and I am lucky to be here in New Zealand studying the sixth form. The former experience has taught me compassion, the latter how to do research. This last one is basically how to read and write. To read and write is not difficult, but to know *how* to do so that is another thing. It amounts to knowing how to think for yourself. Now I know, for instance, that good books always have two themes, and the more profound one is always the one in the background, a more abstract one and never a story. This is why it is always difficult to turn these books into a film where you can only show the less important theme and it is nearly impossible to express the remaining one. I learn all this in my Sixth Form English class here. Night and day I recite lines from Macbeth. Before I came here I had read Julius Caesar, but this is by far better. Not only this, I also study Art History, Mathematics and Music. Now I am a writer, thanks to Mr Lonsdale my English teacher then. I am also a mathematician, thanks to Mr Thompson who taught me the subject. I can not wait to become an artist and a composer which are what the other two of my teachers, the latter of whom is Vicky, had taught me to be. In this book, however, I only talk predominantly about literature.

A British Lanna. ISBN 974–91237–4–3

It is England which turns me into a Christian. This is by no means a compliment to the country, but it certainly is one for the people. At the Moberly Hall where I live I read the translation of Hugo's Les

Misérables, the book which greatly moves me and even before having been through the whole of it had made me profess Christianity. And here is my interpretation now, that God is the Superset. Therefore, if we believe in the one and only one Creator by definition we have Christ in our heart but mathematically speaking we are the Me in Christ in the Universe in God. Here I say Christ as a definition as distinguishable from Jesus who is also a person. A similar definition to the one above, and which is what I prefer now, is this, that it is instead a Me in Universe in Christ in God. It is easy to see from this, that we should explore the universe because they are the same family as ours. In other words they are one of us, though they may not know it, so we need to find and preach the Gospel to them. England and my teachers here have taught me many thing. It is also here that I volunteer in the Community Action and through it got to know Clair. But I had better leave you to read about her yourself.

Edokko no Lanna. ISBN 974–91341–9–2

The word *Edokko* is Japanese. It means 'Edokkite' or 'Tokyoite' in English. *No* is Japanese for 'of'. It also renders a noun preceding it into an adjective, the function which is obviously the case here. The story is set for most parts in Japan, but also contains the excursions I made from Tokyo to New Zealand, Thailand and the US. After having been in Japan for one year, and only after then, I have come to love the country so much. The first instance this happened was when I was travelling on a train across the countryside through the inner part of Kyūshū. Thereafter this has been strengthened by my working as an interpreter in Tokyo and Yokohama, my learning the Japanese archery or Kyūdou, and my having fallen in love with the fermented beans *nattou*.

The Siamese Lanna. ISBN 974–91341–8–4

I really thought when I was writing this book that I was going to die before I could see another book published. This is why I have put into this single volume all the knowledge I have about the Siamese martial arts as well as everything I have learnt from my *saṃnakd́ạḅ* (sword school) Śrị Ȧyudhyạ. I feel that I have already told you everything in this book, so there are nothing more I may add now. You should somehow find the book and read it if you are interested in the Siamese heritage. Because my great-grandmother was an Ayudhyaite, and because my Thai is way better than my Lanna, I identify myself as more a Lanna Siamese (or perhaps a Lannaese Siamese) than a Siamese Lanna despite what the title of the book says. The poem in pages 21 and 22 is quite old and is very dear to me. I have known it since I was a child, even though I do not know who wrote it. I would have liked to think it is very old, dating back to the Ayudhya Empire of Siam, but it can not be that old. Because it calls the name of the country by its new name *Daiy* (Thailand) instead of the name used until the beginning of the 20th century, *Syạm* (Siam), it means that it must be less than a century old. What I plan to do next is to do more research into, and write about this and other arts of fighting in more detail.

D

§ D. Translation

§ D.1 G. L. Dirichlet, 1848

On the reduction of positive quadratic form with three indeterminate integers.

([Lecture in physical- mathematical class meeting of the academy, on 31^{st} *July*, 1848 †]), [by G. L. Dirichlet]
[translated by K N Tiyapan]

It is well known that Lagrange had pointed out for the first time that every binary quadratic form reduces, ie. can transform into another equivalent one the coefficients of which satisfy certain inequality conditions, and at the same time had proven that in every class of positive forms there always exists only one such form, so that in this case the various values of a given determinant corresponding to reduced forms can serve as the representatives of the different classes. Later on after in the "Disquisitiones arithmaticae" the ternary form were looked at from a general point of view did it become necessary for the further development of this theory to extend the study for the positive binary forms by Lagrange to the ternary ones, ie. to find out such inequality conditions between the coefficients that would satisfy one and only form in all classes.

This expansion linked with great difficulties is achieved by *Seeber* in a work specifically devoted to the positive ternary forms, the principal contents of which settles it and which *Gauss* characterises in a most interesting announcement † as follows:

> *We must do full justice to the spirit of the thoroughness by which these facing ‡ us have gone through, and when we for all that have to feel sorry that a great and perhaps much discouragingly complicated nature is attached to it, here the solution of the problem takes 41 pages and the proof 91 pages, thus we will see this by no means as a respected criticism. If a difficult problem or theorem to solve or to prove exists, then the first seeming idea is always to be recognised as a step that a solution or a proof has been found after all, and the question whether this were not of an easier and simpler way would be possible as long as in so doing such a futile question is not considered as of practicability. Therefore we look upon it as untimely to dwell on this question.*

The great complication of Seeberian method has for a longtime stimulated me to set up the theory of reduced ternary forms by a simpler method. As I now allow myself to communicate to the class the result of my effort directed towards this, I think in the interest of briefness and, if I could say so, of the lucidity of the presentation, to have to abide by the geometrical form, in which I have conducted the investigation to which I have laid down as basis the noteworthy relations which occur among the quadratics with two or three elements and with known spatial forms. I begin with the explanation of the outline already given by Gauss in the mentioned announcement on these relations.

§1

The ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy = \varphi, \tag{1}$$

in which we regard x, y, z as first, second, third element, called positive when φ does not become negative for real values of these elements; in one such form the coefficients:

$$a, b, c$$

are always positive, while the coefficient combinations:

$$a'^2 - bc, b'^2 - ac, c'^2 - ab, aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c' = -D, \tag{2}$$

the last $-D$ of which is called the determinant of the form, are negative. † Owing to these conditions there are always three through the equations:

$$\cos\lambda = \frac{a'}{\sqrt{bc}}, \cos\mu = \frac{b'}{\sqrt{ac}}, \cos\nu = \frac{c'}{\sqrt{ab}}$$

fully determined acute or obtuse angles λ, μ, ν, from which a three-edged corner can be built, here the condition necessary to this:

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu - 2\cos\lambda\cos\mu\cos\nu < 1$$

with $D > 0$ coincided. Here nevertheless with the same angles λ, μ, ν two corners symmetrical to each other could be built, therefore we will agree to always choose the corner from these two, with which the edges, as they lie opposite to these angles in sequence, follow one another from left to right with regard to a straight line directed from the vertex 0 to the inside of the corner which can be thought of as going upward. If we now look at the three edges as the positive axes of a coordinate system we could connect the entire infinite space with our form in which we view the product $x\sqrt{a}, y\sqrt{b}, z\sqrt{c}$ as the coordinates of an arbitrary point of the space, and then φ expresses the square of the distance of this point from the vertex, or more general still the square of the distance of two points, the correspondent coordinates of which have those products to differences.

If one establishes now with three new indeterminate elements x', y', z' the linear expressions:

$$x = \alpha x' + \beta y' + \gamma z', y = \alpha' x' + \beta' y' + \gamma' z', z = \alpha'' x' + \beta'' y' + \gamma'' z', \tag{3}$$

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† Crelle's Journal, V. 20, p. 312

† Disquisitiones arithmeticae, art. 271

of which only one restriction shall take place, that the determinant set up from the 9 coefficients α, β, γ, α', β', γ', α'', β'', γ'':

$$\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma'' = E \tag{4}$$

is not zero, then φ changes into a new form φ', with regard to which all correspondings shall be indicated with the accented alphabets. If one lets the new form again correspond to an infinite space, then through it two infinite spaces connect point for point with each other, while every two points correspond to each other when in the expressions of their coordinates:

$$x\sqrt{a}, y\sqrt{b}, z\sqrt{c};\;\; x'\sqrt{a'}, y'\sqrt{b'}, z'\sqrt{c'}$$

the elements x, y, z and x', y', z' are linked with one another through the equation (3). If the expressions just written are the coordinate differences for two pairs of corresponding points, then apparently the same relation among $x, y, \ldots$ still holds, out of which from the above and as a result of $\varphi = \varphi'$ it follows immediately that the distance of every two points of a space is equal to the distance of corresponding ones of another. The two spaces connected with each other are therefore either congruent or symmetrical, ie. they can, while the beginning points 0 and $0'$ are laid on each other, come to such a position that either every point falls on its correspooonding one or on the opposite point of the latter, when we call for short opposite points two points of the same space which lie from the beginning point at the same distance and in the opposite direction. In order to decide which of these two cases takes place, one has lines to draw in the one space from the vertex to three arbitrary points, and then to investigate whether the straight lines drawn in the other from its vertex to the corresponding points present a corresponding series or the opposite one. If one takes for example in the second space the lines from the points with the coordinates:

$$\sqrt{a'}, 0, 0;\;\; 0, \sqrt{b'}, 0;\;\; 0, 0, \sqrt{c'}$$

drawn, lines falling on the positive axes of the second space, then these follow one another from the agreement dealt with above from right to left. For the corresponding points in the first space one has the coordinates:

$$\alpha\sqrt{a'}, \alpha'\sqrt{a'}, \alpha''\sqrt{a'};\;\; \beta\sqrt{b'}, \beta'\sqrt{b'}, \beta''\sqrt{b'};\;\; \gamma\sqrt{c'}, \gamma'\sqrt{c'}, \gamma''\sqrt{c'}.$$

In order to determine whether the lines directed to these points follow one another from left to right, ie. as the axes of the first space, or follow in the reverse order, one can make use of the theorem which is known or easily derivable from known properties ‡, from which the straight lines drawn to the three points $(\xi, \eta, \zeta), (\xi', \eta', \zeta'), (\xi'', \eta'', \zeta'')$ present the same series as the axes of ξ, η, ζ or the opposite one, according to the determinant built from the 9 coordinates, when one gives the term $\xi\eta'\zeta''$ in it the positive sign, is positive or negative. For our case this determinant becomes $E\sqrt{a'b'c'}$; therefore congruence symmetry holds according as E is positive or negative.

Til now the elements x, y, z had arbitrary values. If we let them now only further mean integers, then instead of the integral space we have an infinite system of points parallelly arranged, ie. a point system of which through the intersections of three lines parallel equidistant planes would be created. If we assume now further that the substitution coefficients α, β, γ are also integers and E has the values ± 1, then every integral combination x, y, z would represent an integral combination x', x', z' and vice versa. The parallelepipedal systems thus connected with one another would as a result coincides with the other or with the opposite points of the latter. Yet, here the opposite points of points of one such system again make the same system, the two cases are not different from each other, and this becomes evident also from the circumstance that φ' remains unchanged when one takes α, β, γ with opposite signs through which E changes into $-E$. The two systems are therefore always congruent, and one sees that systems which correspond to two equivalent ternary forms φ and φ' are the same spatial structure in two different patterns. Conversely equivalent forms represent any two different parallelepipedal patterns of the same system. If one takes namely any one point of the system as the common starting point, then one has between the coordinates relation to the two axis systems and therefore also between the elements x, y, z, x', y', z' proportional to them linear equations without constant term, ie. equations of the form (3), and here from our supposition, when x, y, z are integers, x', y', z' must also have the same characteristic and vice versa, therefore it follows that $\alpha, \beta, \gamma, \ldots$ are likewise integers and that $E = \pm 1$. On the other hand one has for the homogeneous entire values of the elements the equation $\varphi = \varphi'$, which accordingly also identically takes place, q.e.d.

Similar interrelations occur between a positive binary form:

$$lx^2 + 2mxy + ny^2$$

and a system of points parallelogrammatically arranged. One takes here two axes leant against each other under the angle θ determined through the equation $m = \sqrt{ln}\cos\theta$, while one always invariably proceed with the discrimination of these axes and for example chooses the second on the left-hand side of the first one, after a fixed side of the plane is denoted as the higher one and $x\sqrt{l}, y\sqrt{l}$ viewed as coordinates, one would obtain a system of points completely determined through the quadratic form, which could be considered as the intersection of two series of equidistant parallel lines. If then between two forms the so-called proper equivalence takes place, so that $\alpha\delta - \beta\gamma$ in the substitution equations $x = \alpha x' + \beta y', y = \gamma x' + \delta y'$ is equal to the positive unity, then the corresponding systems can be brought to the coincidence through movement in the plane, while in the other case where $\alpha\delta - \beta\gamma = -1$, to say in general, one of the systems must be shifted for this purpose.

§2

After we have established in the foregoing the connection between the quadratic forms and certain geometrical patterns, there are a few further properties of these patterns to develop, whereby we would for short call a system of points arranged parallelogrammatically or parallelepipedally a *system of second* or *third order*, and infinite series of equidistant points in straight line a *system of first order*.

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‡ Disquisitiones generales circa superficies curvas auctore, C. F. Gauss §2. VII

It seems that the common character of all three types of the system consists in that when such a system is brought into another position through a movement without rotation, which we wish to know a displacement of, that a point of it changes into the position occupied by another in the beginning, the same happens for all points, therefore that the system in its new position fully coincides with the system in the original one. It can be easily proved that the movability just discussed completely characterises all three types of the systems, and that a system endowed with this characteristic, when it lies in a plane and contains three points not lying in a straight line such that finally a system contains points at least four of which are not found in a plane, will be respectively a system of first, second or third order.

If one has for example a system of points which lie all together in the same straight line, and a and a' are two adjacent points of it, then through a displacement through which a gets to a', a' would get to a'' which is as far from a' as a' is far apart from a; the point a'' therefore also belongs to the system, and the system has no point between a' and a'', here one such point would be known before the movement between a and a'. Here this inspection could be pursued for both sides in the indeterminate, therefore the assertions is proved.

Now let two adjacent points a and a' be in a planar system with the characteristic feature of movability, so that no point of the system is found in the line aa' between a and a'. Here through the displacement from a to a' the infinite straight line aa' moves along by itself, therefore it follows that the entire points of the system in this straight line makes up a system of first order $\ldots{}''a'aaa'a''\ldots$. Here then from the assumption the system still has at least one point outside this straight line, therefore let b be one of the points closest to this straight line. If now enters a displacement through which a gets to b, then the system of first order changes into the new position $\ldots{}''b'bbb'b''\ldots$ and belong in this position to the original system; it is immediately clear that a point of the system can neither be found among the points $\ldots,{}''b,{}'b,b,b',b'',\ldots$ nor among the lines $\ldots{}'bbb'\ldots,\ldots{}'aaa'\ldots$. If one concludes directly, one sees that the entire system can be parallelogrammatically aranged, and that one can choose $aa'b'b$ for a basic parallelogram of it. We further add that through the given construction apparently all parallelogrammatical patterns of which the system is capable could be obtained. It follows from this that the choice of a' up to the obviously necessary restriction that no point lies between a and a' is totally arbitrary, and that b can be taken arbitrarily in the nearest parallel line.

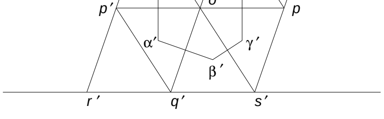
One has finally a system with the characteristic feature of the movability which contains points at least four of which are not lying in the same plane, therefore one lay a plane through any three points of this system not lying in a straight line. Here through any parallel displacement effected with this plane this is moved into itself, consequently points found in this plane build a system of second order from the previous system. As a result one has partitioned this system parallelogrammatically somehow or other, one takes one of the remaining points of the spatial system which lie closest to the plane, and administer a displacement to the system through which an arbitrary point in the plane comes to the point chosen well outside that plane. Through repeated application of this [displacement] and through the movement opposite to it one apparently obtains a parallelepipedal pattern of the given system, and it is immediately clear that the construction specified has the due generality, here the choice of the first plane, the pattern of the system of second order in this plane and finally the choice of points in the neighbouring plane can happen at will.

In the end of this paragraph we will point out further that, as one also partition the same system of second or third order, the parallelogram or parallelepiped lying at the basis of the respective partitioning always retains the same capacity, the geometrical consequence of the sentence is that equivalent forms have the same determinants. If one imagines namely in the plane of a system of second order a line returning to itself, for example a circle line, designates with z the surface area enclosed by it and with s the number of points in the inside of the line, in the course of which it makes no difference whether one wants to include the points on the periphery or not, then obviously the quotient $\frac{z}{s}$ has with growing radius the capacity of a basic parallelogram to the boundary, from which, here s and z are independent of the type of the pattern, the theorem for systems of second order becomes evident. Totally in the same fashion follows the soundness of the assertions for spatial systems.

§3

We will now point out that a system of second order would always admit partitioning by a basic parallelogram, the sides of which are not larger than its diagonals.

I. Let o be an arbitrary point of the system. The remaining points of this system always lie pairwise in the same distance and opposite direction from o. Now let p be one of the points of the pairs, for which the distance from o is smaller than for every other pair. The same smallest distance holds for more than one pair, therefore one would choose p at will in one of these. The given system consists of an infinite quantity of systems of first order congruent among one another and of the same distance, one of which is that for which o and p belong. In one of the two adjacent to this latter one, one takes the point q which is next to o, or, supposing the same shortest distance should occur for two points, arbitrarily takes one of the two. The parallelogram $poqr$ thus obtained has the desired property, here in accordance with the construction $op \leqq oq$, $oq \leqq or$, $oq \leqq os = pq$. A basic parallelogram which satisfies these conditions shall be called a reduced one.



II. We have now the relation between one such parallelogram and the planar system in which it belongs to establish. If $poqr$ is a reduced parallelogram we would be able to, without breaking the generality, assume the angle poq as not obtuse, here in the opposite case the angle at o for the parallelogram adjacent to the same structure is an acute one, and likewise we could assume $op \leqq oq$. Thereupon $or > oq$ is apparent, and we have only the condition $pq \geqq oq$ still to consider. If this is supposed and if we put for the reduction $op = \sqrt{l}$, $oq = \sqrt{n}$, so that consequently $l \leqq n$ the connection of our parallelogram to the entire point system would possibly be described to the effect that the minimum of the distance of any point of the system from

o is equal to $\sqrt{l}$, and that after one has chosen a point at this distance, in all distances still remaining, ie. outside the straight line drawn from o to the former one, the second minimum is equal to $\sqrt{n}$. The precisely stated holds true all in general; what we now add, that namely the first minimum only occurs for the point p (when we always only choose one of two opposite points), the second only for q, holds true with the following exceptions:

1. If $op < oq$, $oq = pq = os$, then the first minimum takes place only for p, the second for q and s.
2. If $op = oq$, $oq < pq = os$, then the minima are equal, and one can exchange p and q with one another.
3. If finally $op = oq = pq = os$, then one can choose one of the points p, q, s as first point and then one of the remainings as second one.

In order to demonstrate the precisely asserted, we have obviously, the opposite points are always equally far from o, only to point out that q lies closer to o, *firstly* than all remaining points in the straight line sqr, with exception of the point s, the distance from o of which according to the assumption is equal to $os = pq \geqq oq$, and *secondly* than all points of the subsequent parallel lines.

Here $pq \geqq op$, $pq \geqq oq$ and the angle poq is not obtuse, therefore the triangle opq and consequently also the oqs congruent with it has no obtuse angle; therefore the perpendicular dropped from o on qs lies between s and q (inclusive), with which one sets:

$$\cos poq = \frac{m}{\sqrt{ln}},$$

where consequently m is not negative, therefore one has:

$$\overline{pq}^2 = l - 2m + n \geqq \overline{oq}^2 = n,$$

and hence:

$$2m \leqq l, 2m \leqq n, 4m^2 \leqq ln.$$

If one further sets the square of the height of our parallelogram ($op = \sqrt{l}$ regarded as base line) equals k, one obtains for the square Δ of its volume:

$$\Delta = lk = ln - m^2 \geqq \frac{3}{4} ln,$$

and hence:

$$\sqrt{k} \geqq \frac{1}{2}\sqrt{3n}.$$

According to this the second line is at least $\sqrt{3n} = oq\sqrt{3}$ away, and the second point is also established.

III. Here the successive minima $\sqrt{l}, \sqrt{n}$ are decided through the system as such and are independent of any fixed pattern and on the other hand, as we have just seen, correspond in quantity with the sides of the reduced parallelogram, therefore one sees that when the system permits various patterns of this fashion, the sides of the reduced parallelograms will always contain the value $\sqrt{l}$ and $\sqrt{n}$. One would essentially obtain as a result all possible basic parallelograms if one draws lines from o to all adjacent points (always with exemption of the opposite points) and then takes the nearest or the two nearest points in one of the respective nearest parallel lines; and here from the definitively demonstrated (II) this nearest or these nearest points lie closer to o than all points of the subsequent parallel lines, therefore one can see from the condition that the second points are to be taken in the first parallel line. Therefore all possible patterns would be produced if one successively connect o with all point pairs for which the successive minima take place, from which at once follows with consideration from (II) that in general and in the second one of the singular cases obtained then there is only one such pattern, in the first and third exception case however there are respectively two and three patterns of the systems.

In our present reference the precisely obtained singular cases correspond with the suppositions $2m = l < n$, $2m < l = n$, $2m = l = n$.

§4

We have so far only dealt with properties of the geometrical structures which is to be looked at from the theory of forms as the constructive representation of well known theorems and are already indicated in the article cited in the introduction. It is now to solve another problem of another kind, the problem namely when a system of second order is given and a fixed point o of it is examined, to determine the part of the plane every point of which lies nearer to o than to any other points of the system. Here the condition that a point does not lie farther from o than from any other v, therein consists that the point with o on the same side of the perpendicular drawn up in the middle of ov, so we would consequently have o to combine with all remaining points of the system and the convex polygon built from all corresponding perpendiculars to construct. But from these perpendiculars in infinite quantity only a limited number comes into question, while the remaining ones do not meet the polygon determined by it. We abide by all suppositions attended to, so that consequently $op \leqq oq$ in the reduced parallelogram ($poqr$), the angle poq is not obtuse and opq, oqp are acute. This supposes, it is easy to understand, that one has only the six vertices p, q, s, p', q', s' of the four parallelograms meeting at o to take into consideration, and that the perpendiculars corresponding to s and s' and the building diagram in the particular case, when poq is a right angle, only touch, which then the same happens for the perpendiculars corresponding to r and r'. If one draw the straight line pq, os, $p'q'$, os', one obtains the congruent triangles:

$$poq, qos, sop', p'oq', q'os', s'op.$$

If one consider only the points p, q, s, p', q', s', one has to draw a perpendicular in the middle of the straight lines going from o to these points, ie the same construction to make as when one wished t find the middle point of circumscribed circles for the designated triangles. Here no obtuse angle is found in the triangles, therefore each two successive perpendiculars not outside the corresponding triangle intersect. One obtains therefore the hexagon $\alpha\beta\gamma\alpha'\beta',\gamma'$ with the centre o and equal opposite angles and sides as the space, inside of which every point is less far apart from o than from one of the points p, q, s, p', q', s', and one is easily convinced that, with exception of r and r', the perpendiculars corresponding to the remaining points do not meet our hexagon. This requires, as a result of symmetry, only for the points in and above the line pop' to be established. For the former ones it is clear; for the latter ones it would hence appear that their distance from o is larger than the diameter of the circle traced around the hexagon. If one designate the square of its radius ρ, then:

$$4\rho\Delta = ln(l-2m+n),$$

from which as a result of $2m \leqq l, 2m \leqq n, \Delta \geqq \frac{3}{4}ln$, it follows:

$$4\rho \leqq \frac{4}{3}(l-2m+n) \leqq \frac{8}{3}n.$$

Here now for the points of the second and the subsequent parallel lines, as already remarked, the square of their distance from o amounts to at least $3n$, therefore there still remain simply the points in $tsqr$ apart from s, q, r to examine. From all of these none is closer to o than t, for which the square of the distance is equal to $4l - 4m + n$, and that this is larger than 4ρ, one immediately sees when one multiplies with Δ and then looks at the inequalities $2m \leqq l \leqq n$. As for the point r, one is also convinced by the same manner that the square of its distance from o is equal to $l + 2m + n > 4\rho$, the only case excluded, where $m = 0$, in which the corresponding perpendicular touches. It is thus demonstrated that every point in the inside of the hexagon $\alpha\beta\gamma\alpha'\beta'\gamma'$, and only one such hexagon, lies closer to the point o than any other of the system. On any side the distance from o would be equal to the distance from a second point, which for example for $\alpha\beta$ is the point q, and every vertex of the diagram is of the same distance from o and another two other points of the the system. The latter statement undergoes a modification only in the special cases when the angle poq is a right angle; thereupon β and γ as well as β' and γ' coincide, and the hexagon turns into a rectangle, of which the corner from o and another three other points of the system are equally far apart.

It goes without saying that one will always obtain the same hexagon whose reduced parallelogram one also lay as foundation of the construction in the singular cases, where more than one exists, just as also that the hexagon or quadrangle corresponding to all the points of the system are congruent and cover the whole plane of it.

We notice further that, as one is easily convinced, the expression:

$$\rho = \frac{ln(l-2m+n)}{4(ln-m^2)}$$

decreases when one therein, assuming l and n constant, allows m to grow from zero up to its limit $\frac{1}{2}l$, so that consequently:

$$\rho \leqq \frac{1}{4}(l+n) \leqq \frac{1}{2}n. \tag{1}$$

Also in addition the following inequality takes place:

$$2\Delta(n-\rho) \geqq ln^2, \tag{2}$$

the soundness of which is immediately evident when one multiplies with 2, moves everything to one side and then applies $\Delta = ln - m^2$, $4\Delta\rho = ln(l-2m+n)$, by the mean of which it changes into $ln(l-2m) + 2mn(n-l) \geqq 0$.

§5

We come now to our true topic and have to prove that every system of third order can be arranged according to a parallelepiped whose faces are reduced parallelograms and whose edges, from which every four are equal to one another, do not exceed their diagonals.

After one has fixed an arbitrary point (0) of the systems, one would choose in pairs of opposite points for which the distance from (0) is a minimum, or when the minimum of the distance exists for several pairs, would arbitrarily choose a point (1) in one of these pairs. From all points outside the straight line (01) one would again choose one of the two nearest (2), through which again the selection under several pairs, for which the same shortest distance takes place, can be arbitrarily made. Here in the whole system, with exception of the points in (01), no point lies closer to (0) than to (2), so the same is valid also for the plane (102), and (102) is a reduced parallelogram for the system which contains this plane (§3, III). One now takes in one of the two nearest parallel planes the point which is closest to (0) or, when the minimum occurs for more than one, one of the nearest ones and connect (0) with the chosen point (3), therefore the parallelepiped would with the edges (01), (02), (03), as is easy to see, suffice the requirement. Next from the construction it follows: (01) $\leqq$ (02) $\leqq$ (03). Here for the bases of the parallelepiped (we would always indicate as such each face opposite to one another in which are found edges two of which do not exceed the third one in size, and the term side faces apply to the four remaining ones) it is already proven that they are reduced, therefore we have in virtue of the precisely noted doubled inequality only to point out further that the four diagonals of the side faces, just as the four diagonals of the body, are not smaller than (03). Now the eight diagonals mentioned above will agree, as one immediately sees, in size with the eight connecting lines which could be drawn from (0) to the eight points lying around (3) in the plane of the higher bases if we indicate this way for convenience the eight vertices of the four parallelograms meeting at (3). The fact that from the afore-mentioned connection lines none is smaller than (03) follows from the condition by which (3) is being chosen.

After we have convinced ourselves that a system of third order can always be partitioned by a reduced parallelepiped, we now have to establish the relation between such parallelepiped and the system and particularly to compare the distance of the point of systems from (0) with one another. We set $(01) = \sqrt{a}, (02) = \sqrt{b}, (03) = \sqrt{c}$ and always hold fast the assumption $a \leqq b \leqq c$.

1. In the plane of the base the conditions discused above (§3, II) occurs, so that consequently the successive minima of the distance are always $\sqrt{a}, \sqrt{b}$ in size, whereby then in the singular cases mentioned there there is an arbitrariness in the choice of the points.

2. We now look at the points outside the plane of the base underneath namely first of all the one in the plane of the base above. Here from the assumption that our parallelepiped is a reduced one, the line (03) is not longer than one of the straight lines drawn from (0) to the eight points lying around (3), so as a result the foot of the perpendicular dropped from (0) on to the plane of the base above would not be farther apart from (3) than from one of the eight points mentioned. This foot point therefore does not fall outside the hexagon or quadrangle constructed to belong to (3) in the last paragraph. Of those eight points can exceptionally, when the foot point falls on one side, one, or it could, when the foot point meets with a vertex, two (three, when the polygon becomes a rectangle,) lie equally close to the foot point as the point (3), while all remaining points of the plane are further apart from that foot point. It follows from this that the shortest distance (amounting to $\sqrt{c}$) from (0) to the a point in the base above is valid in general only for the point (3), but can exceptionally take place for one, two or even three other points.

3. For the consideration of the following parallel planes we have a boundary for the square h of the perpendicular already mentioned to determine. Here the foot point of the perpendicular does not fall outside the hexagon which belongs to (3), therefore, when ρ denotes the square of the radius of the circumscribed circle:

$$h \geqq c - \rho.$$

Now also from §4: $\rho \leqq \frac{1}{2}b \leqq \frac{1}{2}c$, consequently $h \geqq \frac{1}{2}c$. Here therefore the second parallel plane is at least $\sqrt{2c}$ away, therefore there is over the higher base only points, the distance from (0) of which is greater than $\sqrt{c}$.

If one summarises what has been said, one will see that the minimum of the distance for the entire system has the value $\sqrt{a}$, that, after a point is chosen at this distance, the minimum in the still remaining directions amounts to $\sqrt{b}$, and that finally after the second point is also fixed, for all points outside the plane, which is determined through (0) and the first two points, the smallest distance from (0) is reduced to $\sqrt{c}$. If the successive minima $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are also always completely determined in quatity, the same minimum in local relation will not be true without several exceptions which are easy to specify. If for example $a \leqq b$, $b < c$, the first two points are to be chosen in the lower base, whereby the singular cases mentioned in §3, II could occur, while the third point lies in the higher base, has a fixed position there in general, in singular cases however can occupy two, three or four different places. One ever so easily overlook that varieties in the other two cases, where $a < b = c$ or $a = b = c$, could happen.

Here from the assumption of a reduced parallelepiped with the edges $\sqrt{a} \leqq \sqrt{b} \leqq \sqrt{c}$ the length of these edges have yielded themselves as the successive minima of the system, thus it immediately follows that when several reduced parallelepiped exist from which the system can be arranged, these all become in agreement with one another with regard to the lengths of their edges, and it could also be easily pointed out that three of the lines directed from (0) to points of the systems of the lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$ when they only do not lie in the plane, are always the edges of a reduced parallelepiped. It requires therefore only the easy consideration already applied in a similar case (§3, III). Here after this the entire reduced parallelepipeds would be obtained when one construct the successive minima of every possible types, therefore it becomes evident that when this can happen in only one way (to which we also consider the case where, with the equation of two of the quantities $\sqrt{a}, \sqrt{b}, \sqrt{c}$ or with the equation of all three, the three lines are locally completely determined and only an exchange between two or all three can occur) that spatial system would allow only one pattern from a reduced parallelepiped. In all other cases there are several such patterns, the parallelepipeds of which form the basis, which could be either all different from one another or only different in part or even could be all congruent to one another. (Similarly in the two singular cases of a system of second order mentioned above the reduced parallelograms underlying the two or three various patterns were congruent to one another.)

To the determination of the question whether a system of third order permits only one or more than one pattern from a reduced parallelepiped, it would consequently only need the knowledge of a single pattern of the system, and the first case would always and exclusively take place when the reduced parallelepiped given through this pattern is of such a property that all lines which can not be exceeded by others actually exceed this parallelepiped, that is when all diagonals of the faces are larger than their sides and all diagonals of the parallelepiped are similarly larger than the edges of the bodies.

§6

As we now apply the results of the last paragraph to the ternary form, shall the uniformity be assumed because of and in order to avoid pointless differentiation, that one has given every ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy \tag{1}$$

through transposition or change of sign of indeterminate elements, as a result of which the form does not belong to the same class, a single form, that firstly $a \leq b \leq c$ holds, that secondly under the coefficients a', b', c', when not the case that all of them are nonzero and are negative, none has the negative sign, and thirdly, when $b = c$ holds, c' apart from the sign not greater than b', when $a = b$ holds, b' not greater than a', and lastly when $a = b = c$ holds, neither c' greater than b' nor b' greater than a' holds. As is easy to see these condition can only be satisfied in one way and their introduction gives the advantage that, as already without these conditions every ternary form corresponds with a completely determined parallelepiped, now to every parallelepiped also belongs an analytical expression the coefficients of which are also completely determined with regard to their sequence and their signs. This assumed, we mention the form (1) in which

$a \leq b \leq c$ also holds, a reduced one, when it corresponds with a reduced parallelepiped. There the diagonal of the area must not be smaller than the sides themselves, so one has:

$$a \pm 2c' + b \geq b, \;\; a \pm 2b' + c \geq c, \;\; b \pm 2a' + c \geq c.$$

One sets $\sigma = -1$, where a', b', c' are all three negative, otherwise $\sigma = 1$, so these conditions are synonymous with:

$$a \geq 2c'\sigma, \;\; a \geq 2b'\sigma, \;\; b \geq 2a'\sigma, \tag{2}$$

and only, when the equal sign holds, would one of the diagonals in the corresponding parallelogram be equal to a side. The conditions with regard to the diagonals of the parallelepiped result in:

$$a + b + c + 2a'\epsilon + 2b'\delta + 2c'\delta\epsilon \geq c \quad (\delta = \pm 1, \epsilon = \pm 1),$$

where the signs in $\delta = \pm$, $\epsilon = \pm 1$ are arbitrary. One look next at the case where none of the coefficients a', b', c' is negative, and take into account the four sign combinations, as well as that, when a and b are equal to one another then $b' < a'$, so one immediately sees that our inequality is by itself capable of always meet the condition contained, and that the limiting case of the equation in which the diagonals of the edge $\sqrt{c}$ become equal, only once and only then can it occur, when one of the quantities b', c' is equals to zero and when at the same time of the conditions (2) the two quantities b', c' of which are relating to one another, as well as the one which contains a', satisfies the limiting case of the equation. a', b', c' are negative, then our inequality is always fulfilled that the limiting case can not take place, except when $\delta = \epsilon = 1$, so that the consequently the new condition is established:

$$a + b + 2a' + 2b' + 2c' \geq 0, \tag{3}$$

where again the lower sign relates to the equality between a diagonal and the edge $\sqrt{c}$.

The condition just developed (2) and, when a', b', c' are negative, (3) above is are therefore fulfilled, that the inequality takes place in none of the inequalities of the limiting case, therefore in the class to which the form belongs it would not give a second one of these various ones with or without equality signs in the definition condition, here according to at the end of the last paragraph notice that the corresponding system of points can only be partitioned from a reduced parallelepiped. The matter stands differently when the upper signs do not take place in all conditions; there could then occur in the same class several reduced forms that can be derived from a given one. It is sufficient to demonstrate this for a main case. We choose for this the case where $b < c$.

Next one assumes $a > 2c'\sigma$, therefore the direction of the edge $\sqrt{c}$ can only be altered when there are namely in the plane of higher base still one or more points, the distance from the vertex of which amounts to $\sqrt{c}$. When ξ, η are 1 the one such points corresponding values of the element so would, when the third edge depends on it, all the coefficients except a', b' remain unchanged, these respectively change into $a' + c'\xi + b\eta$, $b' + a\xi + c'\eta$ as one is easily and almost convinced without calculation. Now from the specification made earlier are the values of ξ, η which meet the condition, when $a = 2b'\sigma$:

$$\xi = -\sigma, \;\; \eta = 0;$$

when $b = 2a'\sigma$:

$$\xi = 0, \;\; \eta = -\sigma;$$

when simultaneously $a = 2b', b = 2a', c' = 0$:

$$\xi = -1, \;\; \eta = -1;$$

and when a', b', c' are negative and the equation $a + b + 2a' + 2b' + 2c' = 0$ complied with:

$$\xi = 1, \;\; \eta = 1.$$

Corresponding to these four assumptions one has consequently transformed a', b' into:

$$a' - c'\sigma, \;\; b' - a\sigma (= -b'); \;\; -a', \;\; b' - c'\sigma; \;\; -a', \;\; -b'; \;\; a' + b + c', \;\; a + b' + c'.$$

From the third case and generally from the assumption $c' = 0$ one can foresee, here this represents a new form which afterwards one has undertaken in the same one the change of sign stated in the beginning of this paragraph, apparently with the form from which one has derived, become identical. In each of the three remaining assumptions one obtains from application of the necessary sign change a new reduced form belonging to the same class (provided that it does not coincide with the original one), and one obtains two such forms when two of our assumptions hold at the same time. With this then the specification of the form is brought to an end, here apparently the simultaneity of all three assumptions can not take place. If, always under the assumption $b < c$, $a = 2c'\sigma$, one would have, provided that $a < b$, the two edges rotated in the base, for $a = b$ the first edge can also pass into the original position of the two first edges and this new position or both of these new positions of the first two edges must be associated with all the directions of the third one, the original one not excluded.

One can thereby easily remove the inconvenience that in singular cases several reduced forms can be associated in the same class, and thereby take away the exception that for such singular cases in general definition one still include the known secondary conditions which can be proved when one for instance set up the demand that the last coefficient c', provided that it is not fully fixed, maintains the smallest numerical value of which it is capable in the reduced forms of the class, and then likewise with regard to b'. For this to notify an example, we will examine it under the singular cases dealt with earlier where $b < c$, from the three conditions (2) none with the lower signs holds, however the three negative values a', b', c' satisfy the equation:

$$a + b + 2a' + 2b' + 2c' = 0.$$

From the previous observations c' is fixed, and there exists for this case only two reduced forms. a' and b' are the values of the forth and the fifth coefficients in one of them, therefore they are in the other $a'+b+c'$, $a+b'+c'$, or, here the last value are apparently positive, c' is negative and consequently, in order for the sign specification to be sufficient, z to transform into $-z$, rather $-(a'+b+c')$, $-(a+b'+c')$. As it is quite natural, these values suffice when one substitute them for a', b', again the equation:

$$a+b+2a'+2b'+2c'=0,$$

and from them come the values a', b' in the same manner, as they themselves are originated from a', b'. Here according to this the fifth coefficient admits only the two negative values b' and $-(a+b'+c')$, their sum equals $-a-c'$, therefore one see that when one further add to the definition conditions:

$$-b' \leqq \frac{1}{2}(a+c'),$$

the class would contain only one reduced form.

While we conclude the essay, we will still from our principles derive a beautiful theorem, found by *Seeber* through induction and demonstrated by *Gauss* in the announcement already often quoted. From this theorem in a reduced form the production of the first three coefficients is not larger than the doubled absolute value of the determinant.

Here the absolute value of the determinant is equal to the square of the volume of the parallelepiped corresponding to the form, thus consequently, from the expression employed in §5, 3 the inequality to be proved:

$$abc \leqq 2\Delta h,$$

where Δ represents the square of the base. One sets:

$$c=b+t,$$

where consequently t is not negative, draws off from the inequality obtained in §5, 3:

$$h \geqq c-\rho=b-\rho+t,$$

from which one has multiplied it with 2Δ, the equation:

$$abc=ab^2+abt,$$

thus one obtains:

$$2\Delta h-abc \geqq 2\Delta(b-\rho)-ab^2+(2\Delta-ab)t.$$

Here now from the inequality at the end of §4, $2\Delta(b-\rho)-ab^2$ is not negative and $2\Delta-ab \geqq \frac{1}{2}ab$ is positive, therefore the truth of the theorem becomes evident.

§ D.2 G. F. Voronoi, 1908 (I)

New applications of continuous parameters to the theory of the quadratic form
First Memoir
On some properties of the perfect positive quadratic forms
by Mr. Georges Voronoï in Warsaw

[*Journal für die reine und angewandte Mathematik*, V. 133, p. 97–178, 1908]
[translated by K N Tiyapan]

Introduction

Hermite had introduced in the theory of numbers a new and fruitful principle, namely: being given a set (x) of systems $(x_1, x_2, \ldots, x_n)$ for all the values of $x_1, x_2, \ldots, x_n$, one associates with the set (x) a set (R) composed of the domains in a manner such that by studying the set (R) one studies at the same time the set (x) .

Hermite has shown † numerous applications of the new principle to the generalisation of continuous fractions, to the study of algebraic units, etc.

The ideas of Hermite have been developed in the works of Mr.'s Zolotareff, Charve, Selling, Minkowski. ‡

I intend to publish a series of Mémoires in which I shall show new applications of the principle of Hermite to the various problems of the arithmetic theory of definite and indefinite quadratic forms.

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† *Hermite.* Extraits de lettres de M. Ch. Hermite à M. Jacobi sur differents objets de la théorie des nombres. [Excerpts from letters of Mr. Ch. Hermite to Mr. Jacobi on various subjects in the theory of numbers] (This Journal V. 40, p. 261)

Hermite. Sur l'Introduction des variables continues dans la théorie des nombres. [On the introduction of the continuous variables in the theory of numbers] (This Journal V. 41, p. 191)

Hermite. Sur la théorie des formes quadratiques. [On the theory of quadratic forms] (This Journal V. 47, p 313)

‡ *Zolotareff.* On an indeterminate equation of the third degree (Petersbourg, 1869, in Russian.)

Zolotareff. Theory of complex integers with applications to the integral calculus. (Petersbourg, 1874, in Russian.)

Charve. De la réduction des formes quadratiques ternaires positives et de leur application aux irrationelles de troisième degré. [Of the reduction of positive ternary quadratic forms and of their application to the irrationals of third detree] (Suppl. to V. IX of Annales Scientifiques de l'Ecole Normale Supérieure, 1880)

Selling. Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

Minkowski. Geometrie der Zahlen. [Geometry of numbers] (Leipzig, 1896)

In this Mémoire, I study the properties of the minimum of positive quadratic forms and of their various representations by systems of integers.

Hermite has discovered an important property of the minimum M of positive quadratic forms $\sum a_{ij}x_ix_j$ in n variables and of the determinant D, namely:

$$M \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D},$$

and he has demonstrated numerous applications of this formula.

In a letter to Jacobi, Hermite has said §:

"That which precedes sufficiently indicates an infinity of other analogous consequences which, all, will depend on the difficult study of an exact limit of the minimum of any definite form. Thereupon I then form only one conjecture. My first studies in the case of a form in n variables of the determinant D have given me the limit $\left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D}$, I am inclined to presume, but without being able to demonstrate that the numerical coefficient $\left(\frac{4}{3}\right)^{\frac{n-1}{2}}$ has to be replaced by $\frac{2}{\sqrt[n]{n+1}}$"

Mr.'s Korkine and Zolotareff has under taken the study of the exact limit of the minimum of positive quadratic forms of the same determinant.

By indicating with $M(a_{ij})$ the minimum and with $D\ (a_{ij})$ the determinant of the form $\sum a_{ij}x_ix_j$, one will have the minimum

$$\mathcal{M}(a_{ij}) = \frac{M(a_{ij})}{\sqrt[n]{D(a_{ij})}}$$

of a positive quadratic form with determinant 1.

By virtue of the theorem of Hermite the function $\mathcal{M}(a_{ij})$ verifies the inequality ¶

$$\mathcal{M}(a_{ij}) \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}},$$

therefore it is bounded within the set (f) of all the positive quadratic forms of real coefficients.

Mr.'s Korkine and Zolotareff have demonstrated † that the function $\mathcal{M}(a_{ij})$ possesses many maxima in the set (f) which correspond to the various classes of equivalent positive quadratic forms.

The limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite in the letter to Jacobi (source cited) is only a maximum value of the function $\mathcal{M}(a_{ij})$.

The binary and ternary positive quadratic forms possess a single maximum which is therefore, in this case, the exact limit of values of the function $\mathcal{M}(a_{ij})$.

Reckoning from the number of variables $n \geq 4$, one meets many maxima of the function $\mathcal{M}(a_{ij})$.

Mr.'s Korkine and Zolotareff have found many values of various maxima of the function $\mathcal{M}(a_{ij})$ which exceed the limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite, but do not exceed the limit 2.

The study of the exact limit of the minimum of positive quadratic forms of the equal determinant comes down, after Mr.'s Korkine and Zolotareff, to the study of all the various classes of positive quadratic forms to which correspond the maximum values of the function $\mathcal{M}(a_{ij})$.

The maximum maximorum of values of the function $\mathcal{M}(a_{ij})$ is the largest value of the function $\mathcal{M}(a_{ij})$ which presents a numerical function as $\mu(n)$.

Mr.'s Korkine and Zolotareff have determined the following values of the function $\mu(n)$:

$$\mu(2) = \sqrt{\frac{4}{3}},\ \mu(3) = \sqrt[3]{2},\ \mu(4) = \sqrt[4]{4},\ \mu(5) = \sqrt[5]{8},.$$

They have called extreme the quadratic forms which yield to the function $\mathcal{M}(a_{ij})$ a maximum value.

The extreme quadratic forms enjoy an important property, namely:

I. Any extreme quadratic form is determined by the value of its minimum and by all the representations of the minimum.

Mr.'s Korkine and Zolotareff have determined all the classes of extreme forms in 2, 3, 4 and 5 vertices.

By studying these extreme forms, I have observed that they are all well defined by the property (I). There is only reckoning from positive forms in six variable which I have encountered positive quadratic forms which enjoyed the property (I) and are not of extreme forms.

I call "perfect" any positive quadratic form which enjoys the property (I).

I demonstrate that the set of all the perfect forms in n variables can be divided into classes the number of which is finite.

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§ This Journal. V 40, p. 296

¶ Mr. Minkowski has demonstrated an upper limit of the function $\mathcal{M}(a_{ij})$

$$\mathcal{M}(a_{ij}) \leq n$$

much simpler than that from Hermite.

(*Minkowski.* Über die positiven quadratischen Formen und über kettenbruchähnliche Algorithmen. [On the positive quadratic forms and on continued fraction algorithm] This Journal V. 107, p. 291)

† *Korkine* and *Zolotareff.* Sur les formes quadratiques. [On the quadratic forms] Mathematische Annalen, V. VI, p. 366 and V. XI, p. 242

All extreme form being, by virtue of the property I, a perfect form, it results in that the function $\mu(n)$ presents the maximum of values of the function $\mathcal{M}(a_{ij})$ which correspond to the various classes of perfect forms.

I have established an algorithm for the search of various perfect forms by introducing a definition of contiguous perfect forms.

To that effect, I make correspond to the set (φ) of all the perfect forms in n variables a set (R) of domains in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

The set (R) of domains in $\frac{n(n+1)}{2}$ dimensions presents a partition of the set (f) of all the positive quadratic forms in n variables.

Each domain R possesses in the set (R) a contiguous domain which is well determined by any one face in $\frac{n(n+1)}{2} - 1$ dimensions of the domain R.

I demonstrate that the domain R corresponding to the perfect form $\varphi(x_1, x_2, \dots, x_n)$ being determined by the linear inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0, \quad (k = 1, 2, \dots, \sigma)$$

one will have σ perfect forms defined by the equalities

$$\varphi_k(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n) + \rho_k \Psi_k(x_1, x_2, \dots, x_n), \qquad (k = 1, 2, \dots, \sigma) \tag{1}$$

where

$$\Psi_k(x_1, x_2, \dots, x_n) = \sum p_{ij}^{(k)} x_i x_j,$$

provided that the positive parameter ρ_k $(k = 1, 2, \dots, \sigma)$ presents the smallest value of the function

$$\frac{\phi(x_1, x_2, \dots, x_n) - M}{-\Psi(x_1, x_2, \dots, x_n)}$$

where $\Psi(x_1, x_2, \dots, x_n) < 0$ and M is minimum of the form $\phi(x_1, x_2, \dots, x_n)$.

I call "contiguous to the perfect form $\phi(x_1, x_2, \dots, x_n)$" the perfect forms (1).

Any substitution in integer coefficients and with determinant ±1 belonging to the group g of substitutions which do not change the form ϕ permute only the forms (1). One can, therefore, divide the forms (1) into classes of equivalent forms with the help of substitutions of the group g. By choosing one form in each class, one will have a system of perfect forms contiguous to the perfect form ϕ which can replace the system (1).

By proceeding in this manner, one can obtain a system complete of representatives of various classes of perfect forms.

The corresponding domains will form complete system of representatives of various classes of the set (R).

I have remarked that a similar system

$$R, R_1, R_2, \dots, R_{\tau-1} \tag{2}$$

of domains of the set (R) can serve towards the reduction of positive quadratic forms.

I call reduced any positive quadratic form belonging to one of the domains (2).

It results from this definition:

I. Any positive quadratic form can be transformed into an equivalent reduced form, with the help of a substitution which presents a product of substitutions belonging to a series of substitutions

$$S_1, S_2, \dots, S_m$$

which depend only on the choice of the system (2).

II. Two reduced forms can be equivalent only provided that the corresponding substitution belonged to a series of substitutions the number of which is finite.

The weak point of the new method of reduction of positive quadratic forms, demonstrated in this Mémoire, consists in that the number of substitutions which transform into itself the domains of the set (R) or their faces is, in general, very large.

The application of the general theory demonstrated in this Mémoire to the numerical examples will be particularly facilitated if one knew how to solve the following problem:

Being given a group G of substitutions which transform into itself a domain R, one would like to partition this domain into equivalent parts the number of which will be equal to the number of substitutions of the group G and on condition that the number of faces in $\frac{n(n+1)}{2} - 1$ dimensions of domains obtained be the smallest possible.

I show in this Mémoire the solution of the problem introduced in two cases: $n = 2$ and $n = 3$.

From the number of variable $n \geq 4$, I do not know any practical solution of the problem posed.

First Part

General theory of perfect positive quadratic forms and domains which correspond to them.

Definition of perfect quadratic forms.

1 Let

$$\phi(x_1, x_2, \dots, x_n) = \sum a_{ij} x_i x_j \tag{1}$$

be any positive quadratic form. By indicating with

$$(l_{11}, l_{21}, \dots, l_{n1}), (l_{12}, l_{22}, \dots, l_{n2}), \dots, (l_{1s}, l_{2s}, \dots, l_{ns}) \tag{2}$$

the various representations of the minimum M of the form $\sum a_{ij} x_i x_j$, one will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M, \quad (k = 1, 2, \dots, s) \tag{3}$$

One will not consider in the following the two systems

$$(l_{1k}, l_{2k}, \ldots, l_{nk}) \quad \text{and} \quad (-l_{1k}, -l_{2k}, \ldots, -l_{nk}), \quad (k = 1, 2, \ldots, s)$$

as different and one will arbitrarily choose one of these systems.

On the ground of the supposition made, one will have the inequality

$$\sum a_{ij} x_i x_j > M$$

provided that a system $(x_1, x_2, \ldots, x_n)$ of integer values of variables $x_1, x_2, \ldots, x_n$ did not belong to the series (2), excluding the system $x_1 = 0, x_2 = 0, \ldots, x_n = 0$.

By considering the equalities (3) as the equations which serve to determine $\frac{n(n+1)}{2}$ coefficients of the quadratic form $\sum a_{ij} x_i x_j$, one will have only two cases to examine:

1.) there exist a finite number of solutions of equations (3),

2.) the equations (3) admit only a single system of solutions.

2 Let us examine the first case,

Let us suppose that there exists an infinite number of solutions of equations (3).

One will find in this case an infinite number of values of parameters

$$p_{ij} = p_{ji}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

verifying the equations

$$\sum p_{ij} l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, s) \tag{4}$$

the values $p_{ij} = 0,\ i = 1, 2, \ldots, n; j = 1, 2, \ldots, n$ being excluded.

By indicating

$$\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j,$$

let us consider the set of positive quadratic forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho \Psi(x_1, x_2, \ldots, x_n), \tag{5}$$

the parameter ρ being arbitrary.

For a quadratic form determined by the equality (5) to be positive, it is necessary and sufficient that the corresponding value of the parameter ρ be continuous in a certain interval

$$-R' < \rho < R.$$

It can turn out that $R = +\infty$, in this case the lower limit $-R'$ will be finite. By replacing in the equality (5) the form $\Psi(x_1, x_2, \ldots, x_n)$ by the form $-\Psi(x_1, x_2, \ldots, x_n)$, that which is permitted by virtue of (4), one will have the interval

$$-R < \rho < R',$$

therefore one can suppose that the upper limit R is finite.

The corresponding quadratic form, determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n),$$

will not be positive, but it will not have negative values either; one concludes that least for a system $(\xi_1, \xi_2, \ldots, \xi_n)$ of real values of variables $x_1, x_2, \ldots, x_n$ the form $f(x_1, x_2, \ldots, x_n)$ attains in its value the smallest which is zero, and it follows that the system $(\xi_1, \xi_2, \ldots, \xi_n)$ verifies the equation

$$\frac{\partial f}{\partial \xi_i} = \frac{\partial \varphi}{\partial \xi_i} + R\frac{\partial \Psi}{\partial \xi_i} = 0. \quad (i = 1, 2, \ldots, n)$$

By eliminating from these equations $\xi_1, \xi_2, \ldots, \xi_n$ one obtains the equation

$$D(R) = \begin{vmatrix} a_{11} + RP_{11}, & a_{12} + RP_{12}, & \ldots, & a_{1n} + RP_{1n} \\ a_{21} + RP_{21}, & a_{22} + RP_{22}, & \ldots, & a_{2n} + RP_{2n} \\ & \ldots & & \\ a_{n1} + RP_{n1}, & a_{n2} + RP_{n2}, & \ldots, & a_{nn} + RP_{nn} \end{vmatrix} = 0.$$

The smallest positive root of this equation presents the value of R searched for.

3 Let us examine the set (f) of positive quadratic forms determined by the equality (5) with condition

$$0 < \rho < R. \tag{6}$$

Theorem. To the set (f) belongs a quadratic form $\varphi_1(x_1, x_2, \ldots, x_n)$ which is well determined by the following conditions:

1. all the representations of the minimum M of the form $\varphi(x_1,\ x_2, \ldots, x_n)$ are also representations of the minimum M of the form $\varphi_1(x_1, x_2, \ldots, x_n)$,

2. the form $\varphi_1(x_1, x_2, \ldots, x_n)$ moreover possesses at least another representation of the minimum M.

Let us indicate by $M(\rho)$ the minimum and by $D(\rho)$ the determinant of the quadratic form $f(x_1, x_2, \ldots, x_n)$ defined by the equality (5) with condition (6).

By virtue fo the theorem by Hermite, one will have the inequality

$$M(\rho) \leq \mu(n) \sqrt[n]{D(\rho)}. \tag{7}$$

We have demonstrated that $D(R) = 0$, it results in that a value of the parameter ρ can be chosen in the interval (6) such that the inequality

$$\mu(n)\sqrt[n]{D(\rho)} < M$$

holds. One will have, because of (7),

$$M(\rho) < M. \tag{8}$$

Let us indicate by $(l_1, l_2, \ldots, l_n)$ a representation of the minimum $M(\rho)$ of the form $f(x_1, x_2, \ldots, x_n)$ verifying the inequality (8).

One will have

$$\varphi(l_1, l_2, \ldots, l_n) + \rho\Psi(l_1, l_2, \ldots, l_n) < M, \tag{9}$$

and as a result

$$\varphi(l_1, l_2, \ldots, l_n) > M \text{ and } \Psi(l_1, l_2, \ldots, l_n) < 0. \tag{10}$$

This posed, let us find the smallest value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)} \tag{11}$$

determined with condition

$$\Psi(x_1, x_2, \ldots, x_n) \leq 0. \tag{12}$$

To that effect, let us examine the inequality

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)} \leq \frac{\varphi(l_1, l_2, \ldots, l_n) - M}{-\Psi(l_1, l_2, \ldots, l_n)}.$$

By virtue of (9), (10) and (12), one will have

$$\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) < M.$$

The quadratic form $\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n)$ being positive, there exists only a limited number of integer values of $x_1, x_2, \ldots, x_n$ verifying this inequality. Among these systems are found all the systems which give back to the function (11) the smallest value determined with condition (12).

Let us indicate by

$$\left(l_1', l_2', \ldots, l_n'\right), \left(l_1'', l_2'', \ldots, l_n''\right), \ldots, \left(l_1^{(r)}, l_2^{(r)}, \ldots, l_n^{(r)}\right)$$

all the representations of the positive minimum ρ_1 of the function (11).

By declaring

$$\varphi_1(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_1\Psi(x_1, x_2, \ldots, x_n),$$

one obtains the positive quadratic form $\varphi_1(x_1, x_2, \ldots, x_n)$ the minimum M of which is represented by the systems (2) and (13), this is that which one will demonstrate without trouble.

4 With the help of the procedure previously shown, one will determine a series of positive quadratic forms

$$\varphi, \varphi_1, \varphi_2, \cdots \tag{14}$$

which enjoy the following property: by indicating with s_k the number of representations of the minimum of the form $\varphi_k (k = 1, 2, \ldots)$, one will have the inequalities

$$s < s_1 < s_2 < \cdots \tag{15}$$

A similar series of positive quadratic forms of n variables can not be extended indefinitely, this is that which we will demonstrate with the help of the following lemma.

Lemma. The number of various representations of the minimum of a positive quadratic form in n variables does not exceed $2^n - 1$.

Let us indicate by $(l_1, l_2, \ldots, l_n)$ and $(l_1', l_2', \ldots, l_n')$ any two representations of the minimum M of the positive quadratic form $\sum a_{ij}x_ix_j$.

Let us suppose that by declaring

$$l_i' = l_i + 2t_i, \quad (i = 1, 2, \ldots, n) \tag{16}$$

the number $t_1, t_2, \ldots, t_n$ would be integer.

As

$$\sum a_{ij}l_i'l_j' = M \quad \text{and} \quad \sum a_{ij}l_il_j = M,$$

by virtue of (16), it becomes

$$\sum a_{ij}l_it_j + \sum a_{ij}t_it_j = 0$$

One will present this equality under the form

$$\sum a_{ij}(l_i + t_i)(l_j + t_j) + \sum a_{ij}t_it_j = \sum a_{ij}l_il_j. \tag{17}$$

By noticing that

$$\sum a_{ij}t_it_j \geq \sum a_{ij}l_il_j,$$

one finds, by virtue of (17),

$$\sum a_{ij}(l_i + t_i)(l_j + t_j) \leq 0,$$

therefore it is necessary that

$$\sum a_{ij}(l_i + t_i)(l_j + t_j) = 0,$$

and consequently

$$l_i + t_i = 0. \quad (i = 1, 2, \ldots, n)$$

Because of (16), one obtains

$$l_i' = -l_i. \quad (i = 1, 2, \ldots, n)$$

This posed, let us divide the set (X) of all the systems $(x_1, x_2, \ldots, x_n)$ of integer values of $x_1, x_2, \ldots, x_n$ into 2^n classes, with regard to the modulo 2.

We have demonstrated that two different representations of the minimum M of the form $\sum a_{ij}x_ix_j$ will not belong to the same class; neither will any representation of the minimum M belong to the class made up of systems $(x_1, x_2, \ldots, x_n)$ satisfying the conditon

$$x_i \equiv 0(\text{mod}2), \quad (i = 1, 2, \ldots, n)$$

therefore the number of various representations of the minimum of a positive quadratic form can not be greater than $2^n - 1$.

5

We have demonstrated that the series (14) of positive quadratic forms satisfying the condition (15) can not be extended indefinitely, therefore the series (14) will be terminated by a form φ_k which enjoys the following property: the form φ_k is determined by the representations of its minimum.

Definition. One will call perfect any positive quadratic form which is determined by the representations of its minimum.

Let us suppose that the form (1) be perfect, one will have in this case only a single system of solutions of equations (3).

On the ground of the supposition made, the equations

$$\sum p_{ij}l_{ik}l_{jk} = 0, \quad (k = 1, 2, \ldots, s)$$

admit only a single system of solutions

$$p_{ij} = p_{ji} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By effecting the solution of equations (3), one obtains the equalities

$$a_{ij} = \alpha_{ij}M, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where the coefficients α_{ij} are rational.

It results in that the perfect form $\frac{\varphi}{M}$ is of rational coefficients. In the following one will not consider as different the perfect forms of proportional coefficients.

Fundamental properties of perfect quadratic forms.

6

Let

$$\varphi(x_1, x_2, \ldots, n) = \sum a_{ij}x_ix_j$$

be a perfect quadratic form. Let us suppose that all the different representations of the minimum of the perfect form φ make up the series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \tag{1}$$

By choosing any n systems in this series, let us examine the determinant

$$\begin{vmatrix} l_{11}, & l_{12}, & \ldots, l_{1n} \\ l_{21}, & l_{22}, & \ldots, l_{2n} \\ & \ldots & \\ l_{n1}, & l_{n2}, & \ldots, l_{nn} \end{vmatrix} = \pm\omega. \tag{2}$$

All the determinants that one can form this way can not cancel each other out. By supposing the contrary, one will have s equations of the form

$$l_{ik} = \sum_{r=1}^{n-1} l_{ir}u_r^{(k)}, \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, s) \tag{3}$$

One will choose a system of $\frac{n(n+1)}{2}$ parameters $p_{ij} = p_{ji}$ verifying $\frac{n(n+1)}{2}$ equations

$$\sum p_{ij}l_{ir}l_{jt} = 0, \quad (r = 1, 2, \ldots, n-1; t = 1, 2, \ldots, n-1)$$

and by virtue of (3), one will have

$$\sum p_{ij}l_{ik}l_{jk} = 0, \quad (k = 1, 2, \ldots, s)$$

which is impossible.

The numerical value ω of the determinant (2) can not exceed a fixed limit. To demonstrate this, let us effect a transformation of the perfect form φ with the help of a substitution

$$x_i = \sum_{r=1}^{n} l_{ir} x'_r; \quad (i = 1, 2, \ldots, n) \tag{4}$$

one will obtain a form

$$\varphi'(x'_1, x'_2, \ldots, x'_n) = \sum a'_{ij} x'_i x'_j,$$

where

$$a'_{ii} = M. \quad (i = 1, 2, \ldots, n) \tag{5}$$

By indicating with D' the determinant of the form φ', one will have the inequality

$$a'_{11} a'_{22} \cdots a'_{nn} \geq D',$$

by virtue of the known property of positive quadratic forms.

Considering (5), one obtains

$$M^n \geq D'. \tag{6}$$

By indicating with D the determinant of the form φ, one will have, because of (2) and (4),

$$D' = D\omega^2,$$

therefore the inequality (6) reduces to the one here:

$$D\omega^2 \leq M^n.$$

By virtue of the theorem by Hermite, one has the inequality

$$M \leq \mu(n)\sqrt[n]{D};$$

it follows that

$$\omega \leq [\mu(n)]^{\frac{n}{2}}.\dagger \tag{7}$$

7

Any perfect form will obviously be transformed into a form, also perfect, with the help of all linear substitution of integer coefficients and of determinant ±1.

One concludes this that there exists a finite a finite number of equivalent perfect forms.

The set (φ) of all the perfect forms in n variables can be divided into different classes provided that each class be made up of all the equivalent perfect forms.

Theorem. The number of different classes of perfect forms in n variables is finite.

Let us indicate by

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n \quad (k = 1, 2, \ldots, s)$$

s linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s \tag{8}$$

which correspond to the systems (1) of representations of the minimum of the form Ψ.

One establishes this way a uniform correspondence between a perfect form φ and the system (8) of linear forms.

Let us suppose that one had transformed the perfected form φ with the help of a substitution S by integer coefficients and with determinant ±1, one will obtain an equivalent perfect form φ'. Let us indicate by

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{9}$$

the corresponding system of linear forms.

One will easily demonstrate that the substitution T, adjointed to the substitution S §, will transform the system (8) into a system (9).

One concludes that a certain reduction of perfect forms can be effected with the help of the reduction of corresponding systems of linear forms.

The reduction of the system (8) comes down, by virtue of (7), to the reduction of any n linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_n \tag{10}$$

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† See the Mémoire of Mr.'s Korkine and Zolotareff *sur les formes quadratiques positives.* (Mathematische Annalen V. XI, p. 256)

§ The substitution S being defined by the equalities

$$x_i = \sum_{k=1}^{n} \alpha_{ik} x'_k, \quad (i = 1, 2, \ldots, n)$$

one calls "substitution adjointed to S" the substitution T which is determined by the equalities

$$\sum_{k=1}^{n} \alpha_{ik} x_k = x'_i. \quad (i = 1, 2, \ldots, n)$$

belonging to the system (8) and with determinant $\pm\omega$ which does not cancel each other out.

One will determine with the help of the known method a substitution T which will transform the linear forms (10) of integer coefficients into linear forms,

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_n \tag{11}$$

satisfying to the following conditions

$$\begin{cases} \lambda'_k = p_{k,k}x' + p_{k+1,k}x'_{k+1} + \ldots + p_{n,k}x'_n, \quad (k = 1, 2, \ldots, n) \\ p_{11}p_{22}\cdots p_{nn} = \omega \text{ and } p_{kk} > 0, \quad (k = 1, 2, \ldots, n) \\ 0 \leq p_{k+i,k} < p_{kk}. \quad (i = 1, 2, \ldots, n-k; k = 1, 2, \ldots, n) \end{cases}$$

The coefficients of forms (11) being integers, as a result they do not exceed fixed limits.

The substitution T will transform the system (8) into a system

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{12}$$

of linear forms. By examining successively the determinants of forms

$$(\lambda'_k, \lambda'_2, \ldots, \lambda'_n), (\lambda'_1, \lambda'_k, \ldots, \lambda'_n), \ldots, (\lambda'_1, \lambda'_2, \ldots, \lambda'_k), \; (k = n+1, n+2, \ldots, s)$$

one will demonstrate that the numerical values of coefficients of all the linear forms (12) do not exceed fixed limits.

The number of similar systems of linear forms in integer coefficients being limited, it results in that the number of different classes of perfect forms is also limited.

On the domains determined with the help of linear inequalities

8 We have seen in Number 7 that the study of perfect forms can be brought back to the study of certain systems of linear forms.

One will acquire a new basis to these studies by making correspond to each perfect quadratic form in n variables a domain in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

One will address first the general problem by studying the properties of domains determined with the help of linear inequalities. ‡

Let us consider a system of linear inequalities

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

in any real coefficients.

One will call point (x) any system $(x_1, x_2, \ldots, x_m)$ of real values of variables $x_1, x_2, \ldots, x_m$ and one will indicate

$$y_k(x) = p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m. \quad (k = 1, 2, \ldots, \sigma)$$

One will call "domain" the set R of points verifying the inequalities

$$y_k(x) \geq 0. \; (k = 1, 2, \ldots, \sigma) \tag{1}$$

Let us suppose that to the domain R belonged to points verifying the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

one will call such points interior to the domain R, and the domain R will be said to be of m dimensions.

It can be the case that the domain R does not possess interior points. One will demonstrate in this case all the points belonging to the domain R verify at least one equation

$$y_k(x) = 0,$$

the indice h being a value $1, 2, \ldots, \sigma$.

It is important to have a criteria with the help of which one could recognise whether a domain determined by the help of inequalities (1) will be in m dimensions or not.

Fundamental principle. For a domain determined with the help of inequalities (1) to be of m dimensions, it is necessary and sufficient that the equation

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \tag{Ξ}$$

did not reduce into an identity so long as the parameters $\rho_2, \rho_2, \ldots, \rho_\sigma$ *are positive or zero, the values* $\rho_1 = 0, \rho_2 = 0, \ldots, \rho_\sigma = 0$ *being excluded.*

The principle introduced, considered from a certain point of view, is evident, but one arrive at the rigorous demonstration of this principle only with the help of the in depth study of domains determined with the help of linear inequalities.

For more simplicity, one will examine in that which follows only domains satisfying the following conditions: the equations

$$y_k(x) = 0 \; (k = 1, 2, \ldots, \sigma) \tag{2}$$

can not be verified by any point, the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.

It is easy to demonstrate that the general case will always come down to the case examined.

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‡ See: *Minkowski.* Geometrie der Zahlen [Geometry of the numbers], No. 19, p. 39.

Definition. One will call edge of the domain R determined with the help of inequalities (1) the set of points belonging to the domain R and verifying the equations

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, r \text{ where } r < \sigma)$$

provided that these equations defined the values of $x_1, x_2, \ldots, x_m$ *to an immediate common factor.*

By indicating with $(\xi_1, \xi_2, \ldots, \xi_m)$ a point of the edge considered, one will determine all the points of the edge with the help of equalities

$$x_i = \rho\xi_i, \quad (i = 1, 2, \ldots, n)$$

ρ being an arbitrary positive parameter.

This results in that each edge of the domain R is well determined by any point belonging to it.

Let us suppose that the domain R possesses s edges characterised by the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}). \quad (k = 1, 2, \ldots, s)$$

By declaring

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik}, \quad (i = 1, 2, \ldots, m) \tag{3}$$

where

$$\rho_k \geq 0, \quad (k = 1, 2, \ldots, s) \tag{4}$$

one obtains a point (x) belonging to the domain R, the positive or zero parameters $\rho_1, \rho_2, \ldots, \rho_s$ being arbitrary.

10 *Fundamental theorem. Let us suppose that the inequalities (1) which define the domain R satisfy the condition (Ξ).*

The domain R will be of m dimensions and each point belonging to it will be determined by the equalities (3) with condition (4).

The theorem introduced is well known in the case $m = 2$ and $m = 3$.

We will demonstrate that by supposing that the theorem be true in the case of $m - 1$ variables, the theorem will again be true in the case of m variables.

Let us examine first the various inequalities of the system (1). It can be the case that many among them could be put under the form

$$y_h(x) = \sum_{k=1}^{s} \rho_k^{(h)} y_k(x) \quad \text{where} \quad \rho_k^{(h)} \geq 0. \quad (k = 1, 2, \ldots, s; \rho_h^{(h)} = 0)$$

One will call such inequalities dependent and one will exclude them from the system (1).

Let us suppose that the system (1) contained only independent inequalities.

Their number ρ, on the ground of the supposition (2) made, will not be less than m.

This posed, let us examine a set P_h of points belonging to the domain R and verifying an equation

$$y_h(x) = 0, \tag{5}$$

the indice h having a value $1, 2, \ldots, \sigma$.

One will call "face of the domain R" the domain P_h.

On the ground of the supposition made, the face P_h will be in $m - 1$ dimensions.

To demonstrate this, let us make correspond to any point (x) verifying the equation (5) a point (u) in $m - 1$ coordinates $(u_1, u_2, \ldots, u_{m-1})$ by declaring

$$x_i = \sum_{j=1}^{m-1} \alpha_{ij} u_j. \quad (i = 1, 2, \ldots, m)$$

The system of inequalities (1) will be transformed into a system

$$\eta_k(u) \geq 0 \quad (k = 1, 2, \ldots, \sigma; k \neq h) \tag{7}$$

of inequalities in $m - 1$ variables $u_1, u_2, \ldots, u_{m-1}$.

Let us suppose that one knew how to reduce the equation

$$\sum_{k=1}^{\sigma} \rho_k \eta_k(u) = 0 \text{ where } \rho_h = 0 \text{ and } \rho_k \geq 0 \quad (k = 1, 2, \ldots, \sigma) \tag{8}$$

into an identity. By virtue of (6), one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = \rho y_h(x) \quad \text{where} \quad \rho_h = 0.$$

One can not suppose that $\rho > 0$, since otherwise the inequality

$$y_h(x) \geq 0$$

would be dependent and on the ground of the supposition made would not belong to the system (1).

By supposing that $\rho \leq 0$, one will admit $\rho_h = -\rho$ and one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

which is contrary to the hypothesis.

We have supposed that the theorem introduced be true in the case of $m-1$ variables. As the equation (8) can not be reduced into an identity, one concludes that the system of inequalities (7) defines a domain $\mathcal{B}_h$ in $m-1$ dimensions. Moreover, by indicating with

$$(u_{11}, u_{21}, \ldots, u_{m-1,1}), (u_{12}, u_{22}, \ldots, u_{m-1,2}), \ldots, (u_{1t}, u_{2t}, \ldots, u_{m-1,t}) \tag{9}$$

the points which characterised t edges of the domain $\mathcal{B}_h$, one will determine any point (u) of this domain by the equalities

$$u_i = \sum_{k=1}^{t} \rho_k u_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m-1) \tag{10}$$

One will make correspond to the points (9) the points

$$(\xi_r) = (\xi_{1r}, \xi_{2r}, \ldots, \xi_{mr}), \quad (r = 1, 2, \ldots, t) \tag{11}$$

by determining them with the help of equalities (6) and (9).

The points obtained (11) characterise t edges of the domain R belonging to the face P_h. Any point (x) belonging to the face P_h will be determined, on the grounds of (6) and (10), by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{12}$$

Let us notice that all the points (11) verify the equation

$$y_h(x) = 0 \tag{13}$$

and satisfy the conditions

$$y_k(x) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One obtains thus the equalities

$$y_h(\xi_r) \geq 0 \quad (r = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma) \tag{14}$$

The face P_h being in $m-1$ dimensions, the equalities (14) would define the coefficients of the equation (13) to a close by common factor.

11 Let us suppose that one had determined this way all the faces

$$P_1, P_2, \ldots, P_\sigma \tag{15}$$

in $m-1$ dimensions of the domain R.

Let us suppose that the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}) \quad (k = 1, 2, \ldots, s) \tag{16}$$

characterise the various edges of the domain R belonging to the various faces (15).

By indicating

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m) \tag{17}$$

one obtains a set of points which all belong to the domain R.

I say that any point (x) belonging to the domain R can be determined with the help of equalities (17).

One can suppose that the point (x) does not belong to any one of the faces (15), since any point belonging to them can be determined with the help of equalities (12).

By supposing that one had the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

let us arbitrarily choose a point (ξ_r) among those of the series (16) and let us admit

$$x_i' = x_i - \rho \xi_{ir} \text{ where } \rho > 0. \quad (i = 1, 2, \ldots, m) \tag{18}$$

So long as the parameter ρ is sufficiently small, one will also have

$$y_k(x') > 0. \quad (k = 1, 2, \ldots, \sigma)$$

By making the parameter increase in a continuous manner, one will determine with the help of equalities (18) a point (x') verifying an equation

$$y_h(x') = 0$$

and satisfying the condition

$$y_k(x') \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

The point obtained (x') belongs to the face P_h, therefore one can declare

$$x'_i = \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho'_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (18), it becomes

$$x_i = \rho\xi_{ir} + \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho > 0,\ \rho'_k \geq 0. \ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

It remains to demonstrate that the domain F is in m dimensions.

Let us notice that all the points determined by the equalities (17) with condition

$$\rho_k > 0 \quad (k = 1, 2, \ldots, s)$$

are interior to the domain R.

In effect, all the points (16) verify the inequalities

$$y_h(\xi_k) \geq 0. \ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma) \tag{19}$$

By multiplying these inequalities by ρ_k, let us make the sum of inequalities obtained; one will have, because of (17),

$$y_h(x) = \sum_{k=1}^{s} \rho_k y_h(\xi_k) \geq 0. \quad (h = 1, 2, \ldots, \sigma)$$

By virtue of (19), one will have the inequality

$$y_h(x) > 0, \quad (h = 1, 2, \ldots, s) \tag{20}$$

so long as the numbers $y_h(\xi_1), y_h(\xi_2), \ldots, y_h(\xi_s)$ do not cancel each other out.

One can not suppose that the equalities

$$y_h(\xi_k) = 0 \quad (k = 1, 2, \ldots, s)$$

holds, because otherwise all the equations

$$y_1(x) = 0, y_2(x) = 0, \ldots, y_\sigma(x) = 0$$

would be of proportional coefficients, which is contrary to the hypothesis; therefore one will have the inequalities (20), and it follows that the domain R is of m dimensions.

We have demonstrated that the condition (Ξ) is sufficient for the domain R to be of m dimensions. It is easy to demonstrate that this condition is necessary.

12

We have defined in Number 10 the faces in $m - 1$ dimensions of the domain R. This definition can be generalised.

Definition. One will call face in μ dimensions of the domain R $(\mu = 1, 2, \ldots, m - 1)$ a domain $P(\mu)$ formed from points belonging to the domain R and verifying a system of equations

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, \tau) \tag{21}$$

provided that these equations define a domain in μ dimensions composed of points which, all, do not verify any other equation $y_{\tau+1}(x) = 0, \ldots, y_\sigma(x) = 0$.

Let us choose among the points (16) all those which verify the equations (21).

By indicating with

$$\xi_k = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}), \quad (k = 1, 2, \ldots, t)$$

one will declare

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{22}$$

It is easy to demonstrate that any point (x) belonging to the face $P(\mu)$ can be determined with the help of equalities (22).

Corollary. Each face of the domain R is a set of points determined by the equalities (22) provided that any point belonging to it could not be determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

unless all the parameters $\rho_{t+1}, \rho_{t+2}, \ldots, \rho_s$ do not cancel each other.

13

Any point belonging to the domain R either is interior to the domain R or is interior to a face of that domain.

Let us suppose that the point (x) be interior to a face $P(\mu)$ of the domain R which is formed from all the points determined by the equalities (22).

I argue that one can always determine the point (x) by the equalities (22) provided that

$$\rho_k > 0. \quad (k = 1, 2, \ldots, t)$$

To demonstrate this, let us indicate

$$\rho_i = \sum_{k=1}^{t} \xi_{ik}. \quad (i = 1, 2, \ldots, m)$$

The point (α) is interior to the face $P(\mu)$.
By admitting

$$x'_i = x_i - \rho\alpha_i \text{ where; } \rho > 0, \quad (i = 1, 2, \ldots, m) \tag{23}$$

one obtains a point (x'_i) which will be interior to the face $P(\mu)$ so long as the parameter ρ will be sufficiently small; it follows that

$$x'_i = \sum_{k=1}^{t} \rho'_k \xi_{ik} \quad \text{where} \quad \rho'_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (23), one obtains

$$x_i = \sum_{k=1}^{t} (\rho + \rho'_k)\xi_{ik}, \quad (i = 1, 2, \ldots, m)$$

and by making

$$\rho + \rho'_k = \rho_k, \quad (k = 1, 2, \ldots, t)$$

one will have

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k > 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

Let us notice that by making $\mu = m$ and $t = s$, one will indicate with the symbol $P(m)$ the domain R; one concludes that any point (x) which is interior to the domain R can be determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k > 0. \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

On the correlative domains.

14 *Definition. Let us suppose that a domain R be determined with the help of inequalities*

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call correlative to the domain R the domain $\mathcal{R}$ which is formed from all the points (x) determined by the equalities

$$x_i = \sum_{k=1}^{\sigma} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, m) \tag{1}$$

I say that the domain $\mathcal{R}$ will be in m dimensions, if the domain R does not possess points verifying the equations

$$p_{1k}x_1 + p_{1k}x_2 + p_{mk}x_m = 0, \quad (k = 1, 2, \ldots, \sigma)$$

the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.
In effect, if all the points of the domain $\mathcal{R}$ verified the same equation

$$\xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_m x_m = 0,$$

one would have the equalities

$$\xi_1 p_{1k} + \xi_2 p_{2k} + \ldots + \xi_m p_{mk} = 0, \quad (k = 1, 2, \ldots, \sigma)$$

by virtue of (1), which is contrary to the hypothesis.

Theorem. By supposing that the domain R be formed from all the points (x) determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \ (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m) \tag{2}$$

one will define the correlative domain $\mathcal{R}$ with the help of inequalities.

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m \geq 0. \ (k = 1, 2, \ldots, s) \tag{3}$$

Let us indicate by $\mathcal{R}'$ thedomain determined with the help of inequalities (3).
On the ground of the supposition made, all the points

$$(\xi_{11}, \xi_{21}, \ldots, \xi_{m1}), (\xi_{12}, \xi_{22}, \ldots, \xi_{m2}), \ldots, (\xi_{1s}, \xi_{2s}, \ldots, \xi_{ms})$$

characterise the edges of the domain R, and one will have the inequalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \ldots + p_{mh}\xi_{mk} \quad (h = 1, 2, \ldots, \sigma; h = 1, 2, \ldots, \sigma). \tag{4}$$

We have seen in Number 10 that each face P_h in $m-1$ dimensions of the domain R is characterised by the points

$$(\xi_{11}, \xi_{21}, \ldots, \xi_{m1}), (\xi_{12}, \xi_{22}, \ldots, \xi_{m2}), \ldots, (\xi_{1t}, \xi_{2t}, \ldots, \xi_{mt})$$

which verify the equation

$$y_h^x = 0 \tag{5}$$

of the face P_h. One obtains the equalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \ldots + p_{mh}\xi_{mk} = 0 \quad (k = 1, 2, \ldots, t)$$

which define the coefficients $p_{1h}, p_{2h}, \ldots, p_{mh}$ of the equation (5) to an immediate common factor.

One concludes, by virtue of the definition established in Number 9, that the point $(p_{1h}, p_{2h}, \ldots, p_{mh})$ characterises an edge of the domain $\mathcal{R}'$.

By attributing with the indice h the values $1, 2, \ldots, \sigma$, one obtains a series

$$(p_{11}, p_{21}, \ldots, p_{m1}), (p_{12}, p_{22}, \ldots, p_{m2}), \ldots, (p_{1\sigma}, p_{2\sigma}, \ldots, p_{m\sigma})$$

of points which characterise different edges of the domain $\mathcal{R}'$.

I argue that the domain $\mathcal{R}'$ does not possess other edges. To demonstrate this, let us suppose that a point $p_1, p_2, \ldots, p_m$ characterises an edge of the domain $\mathcal{R}'$.

One will have the equalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \ldots + p_m\xi_{mh} = 0, \quad (h = 1, 2, \ldots, t) \tag{6}$$

which define the coefficients $p_1, p_2, \ldots, p_m$ to a nearby common factor, and one will have the inequalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \ldots + p_m\xi_{mh} \geq 0. \quad (k = 1, 2, \ldots, s) \tag{7}$$

Let (x) be any point of the domain R. One will determine the point (x) with the help of equalities (2). By multiplying the inequalities (7) with ρ_k and by making the sum of inequalities obtained, one will have, because of (2),

$$p_1x_1 + p_2x_2 + \ldots + p_mx_m \geq 0.$$

One concludes that the inequalities

$$-p_1x_1 - p_2x_2 - \ldots - p_mx_m \geq 0 \text{ and } p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0,$$

define a domain which is not in m dimensions.

By virtue of the fundamental theorem of Number 10, one will determine in this case positive values or zeros of parameters $\rho, \rho_1, \ldots, \rho_\sigma$ which reduce the equation

$$-\rho(p_1x_1 + p_2x_2 + \ldots + p_mx_m) + \sum_{k=1}^{\sigma} \rho_k(p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m) = 0$$

into an identity.

It follows that

$$p_i = \sum_{k=1}^{\sigma} \frac{\rho_k}{\rho} p_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0. \quad (k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, m)$$

By substituting (6), one will have

$$\sum_{k=1}^{\sigma} \frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk}) = 0. \quad (h = 1, 2, \ldots, t)$$

By virtue of (4), one finds

$$\frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk}) = 0. \ (h = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma)$$

Let us suppose that $\frac{\rho_k}{\rho} > 0$, then

$$\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \ldots + \xi_{mh}p_{mk} = 0, \quad (h = 1, 2, \ldots, t)$$

therefore the coefficients $p_1, p_2, \ldots, p_m$, by virtue of (6), are proportional to the coefficients $p_{1k}, p_{2k}, \ldots, p_{mk}$; it follows that the points $(p_{1k}, p_{2k}, \ldots, p_{mk})$ and $(p_1, p_2, \ldots, p_m)$ characterise the same edge of the domain $\mathcal{R}'$.

By virtue of the fundamental theorem in Number 10, all the points of the domain $\mathcal{R}'$ will be determined by the equality (1), this results in that the domains $\mathcal{R}$ and $\mathcal{R}'$ coincide.

Corollary. Let us suppose that a face $P(\mu)$ in μ dimensions of the domain R be determined by the equations

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m = 0, \quad (k = 1, 2, \ldots, \tau)$$

and that any point (x) belonging to the face $P(\mu)$ be determined by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k\xi_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

The correlative domain $\mathcal{R}$ *will possess a corresponding face* $\mathcal{B}(m-\mu)$ *in* $m-\mu$ *dimensions determined by the equations*

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m = 0 \quad (k = 1, 2, \ldots, t)$$

and any point (x) *belonging to the face* $\mathcal{B}(m-\mu)$ *will be determined by the equalities*

$$x_i = \sum_{k=1}^{\tau} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, \tau; i = 1, 2, \ldots, m)$$

Definition of domains of quadratic forms corresponding to the various perfect forms

15 Let us consider any one perfect quadratic form φ.

Let us suppose that all the representations of the minimum of the form φ make up the series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \tag{1}$$

By indicating

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n, \quad (k = 1, 2, \ldots, s) \tag{2}$$

one corresponds to the series (1) a series of linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s.$$

Let us consider a domain R of quadratic forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{s} \rho_k \lambda_k^2$$

with condition that

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, s)$$

One will say that the domain R correspond to the perfect form φ.

Let us notice that the domain R is in $\frac{n(n+1)}{2}$ dimensions.

By supposing the contrary let us suppose that all the quadratic forms belonging to the domain R verifies a linear equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0.$$

On the ground of the established definition, one will have the equalities

$$\Psi(\lambda_k^2) = 0 \quad (k = 1, 2, \ldots, s)$$

or, that which comes to the same thing, because of (2),

$$p_{ij} l_{ik} l_{jk} = 0 \quad (k = 1, 2, \ldots, s)$$

which is impossible, the form φ being perfect.

On the ground of what has been said in Number 9–14, the domain R possesses s edges characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_s^2. \tag{3}$$

Let us suppose that one had determined all the faces

$$P_1, P_2, \ldots, P_\sigma$$

in $\frac{n(n+1)}{2} - 1$ dimensions of the domain R.

Each face P_k can be determined by two methods:

1. All the quadratic forms belonging to the face P_k verify an equation

$$\Psi_k(f) = \sum p_{ij}^{(k)} a_{ij} = 0$$

which can be determined in such a way that the inequality

$$\Psi_k(f) > 0$$

held so long as the form f belonging to the domain R is exterior to the face P_k.

2. By choosing among the quadratic forms (3) these

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2$$

which verify the equation (4), one will determine all the quadratic forms belonging to the face P_k by the equalities

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2,$$

where

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, t)$$

By virtue of the theorem of Number 14, the domain R can be considered as a set of points verifying the inequalities

$$\Psi_k(f) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

On the extreme quadratic forms

16 Let us indicate by $M(a_{ij})$ the minimum and by $D(a_{ij})$ the determinant of a positive quadratic form $\sum a_{ij}x_ix_j$. The positive quadratic form $\frac{1}{\sqrt[n]{D(a_{ij})}}\sum a_{ij}x_ix_j$ will be of determinant 1 and will possess the minimum

$$\frac{M(\alpha_{ij})}{\sqrt[n]{D(a_{ij})}} = \mathcal{M}(a_{ij}).$$

Let us examine the various value of the function $\mathcal{M}(a_{ij})$ which is well determined in the set (f) of all the positive quadratic forms in n variables.

Definition. One will call extreme [‡] *a positive quadratic form* $\sum a_{ij}x_ix_j$ *which enjoys the property that the corresponding value of the function* $\mathcal{M}(a_{ij})$ *is minimum.*

Let us notice that the function $\mathcal{M}(a_{ij})$ does not change its value when one replaces quadratic form $\sum a_{ij}x_ix_j$ by a form of proportional coefficients.

By attributing to the coefficients of the extreme form $\sum a_{ij}x_ix_j$ variations

$$\epsilon_{ij} = \epsilon_{ji} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

satisfying the condition

$$|\epsilon_{ij}| < \epsilon, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{1}$$

ϵ being an arbitrary positive parameter, let us examine the corresponding value of the function $\mathcal{M}(a_{ij})$.

On the ground of the definition established, one can determine the parameter ϵ such that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) < \mathcal{M}(a_{ij}) \tag{2}$$

held with condition (1) and so long as the coefficients ϵ_{ij} are not proportional to the coefficients

$$a_{ij}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

17 *Theorem. For a quadratic form* $\sum a_{ij}x_ix_j$ *to be extreme, it is necessary and sufficient that it be perfect and that its adjointed form* $\sum \frac{\partial D(a_{ij})}{\partial a_{ij}}x_ix_j$ *be interior to the domain corresponding to the form* $\sum a_{ij}x_ix_j$.

Let us indicate by

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}) \tag{3}$$

the various representations of the minimum $M(a_{ij})$ of the form $\sum a_{ij}x_ix_j$.

Let us consider a quadratic form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$, the parameter ρ being arbitrary. One can determine an interval

$$-\delta < \rho < \delta \text{ where } 0 < \delta < 1 \tag{4}$$

such that all the representations of the minimum of the form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ are found among the systems (3) so long as the variations ϵ_{ij} satisfy the condition (1).

By indicating with

$$M' = \sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk} \text{ and } M = \sum a_{ij}l_{ik}l_{jk} \tag{5}$$

the minima of forms $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ and $\sum a_{ij})x_ix_j$ and with D' and D their determinants, one will have

$$\mathcal{M}(a_{ij} + \rho\epsilon_{ij}) = \frac{\sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}}, \quad \mathcal{M}(a_{ij}) = \frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}.$$

By virtue of (2), one obtains the inequality

$$\frac{\sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}} < \frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}$$

or, that which comes to the same thing,

$$\rho\sum \epsilon_{ij}l_{ik}l_{jk} < M\left(\sqrt[n]{\frac{D'}{D}} - 1\right). \tag{6}$$

This declared, let us suppose that the form $\sum a_{ij}x_ix_j$ be not perfect.

One will determine in this case the variations ϵ_{ij} such that the equalities

$$\sum \epsilon_{ij}l_{ik}l_{jk} = 0,$$

held. By virtue of (6), one will obtain the inequality

$$D' > D.$$

[‡] See the Mémoire of Mr.'s Korkine and Zolotareff, Sur les formes quadratiques [On the quadratic forms], Mathematische Annalen V. VI, p. 368

By developping the determinant D' into a series, one will have the inequality

$$\rho\sum \epsilon_{ij}\frac{\partial D}{\partial a_{ij}} + \frac{\rho^2}{2}\sum \epsilon_{ij}\epsilon_{kh}\frac{\partial^2 D}{\partial a_{ij}\partial a_{kh}} + \ldots > 0. \tag{7}$$

The parameter ρ being arbitrary satisfying the condition (4), it is necessary tht

$$\sum \epsilon_{ij}\frac{\partial D}{\partial a_{ij}} = 0.$$

Mr.'s Korkine and Zolotareff have demonstrated ‡ that in this case one will always have the inequality

$$\sum \epsilon_{ij}\epsilon_{kh}\frac{\partial^2 D}{\partial a_{ij}\partial a_{kh}} < 0,$$

therefore the inequality (7) is impossible.

18 We have demonstrated that the form $\varphi = \sum a_{ij}x_ix_j$ has to be perfect.

Let us suppose that the domain R corresponding to the perfect form φ be determined by σ inequalities

$$\Psi_r(f) = \sum p_{ij}^{(r)}a_{ij} \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

On the grounds of these inequalities, one will have

$$\Psi_r(\lambda_k^2) = \sum p_{ij}^{(r)}l_{ik}l_{jk} \geq 0. \ \ (k = 1, 2, \ldots, s; r = 1, 2, \ldots, \sigma) \tag{8}$$

Let us declare

$$\epsilon_{ij} = tp_{ij}^{(r)} \ \text{ where } \ t > 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By virtue of (6), one will have

$$\rho t\sum p_{ij}^{(r)}l_{ik}l_{jk} < M\left(\sqrt[n]{\frac{D'}{D}} - 1\right). \tag{9}$$

Let us attribute to the parameter ρ a positive value satisfying the condition (4), by virtue of (8) and (9) there will arrive

$$D' > D.$$

By developing the determinant D' into a series, one obtains the inequality

$$\rho t\sum p_{ij}^{(r)}\frac{\partial D}{\partial a_{ij}} + \frac{(\rho t)^2}{2}\sum p_{ij}^{(r)}p_{ih}^{(r)}\frac{\partial^2 D}{\partial a_{ij}\partial a_{kh}} + \ldots > 0.$$

The positive parameter ρ being as small as one wish, it follows that

$$\sum p_{ij}^{(r)}\frac{\partial D}{\partial a_{ij}} > 0. \quad (r = 1, 2, \ldots, \sigma)$$

It is thus demonstrated that the form $\sum \frac{\partial D}{\partial a_{ij}}x_ix_j$, adjointed to the form φ, is interior to the domain R.

I argue that in this case the perfect form φ will be extreme.

By supposing the contrary, let us suppose that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) \geq \mathcal{M}(a_{ij}) \tag{10}$$

be verified by any one system of variations ϵ_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ satisfying the condition (1) however small the parameter ϵ may be.

By virtue of (10), one obtains

$$\sum \epsilon_{ij}l_{ik}l_{jk} \geq M\left(\sqrt[n]{\frac{D'}{D}} - 1\right); \ \ (k = 1, 2, \ldots, s) \tag{11}$$

the inequality obtained has to hold whatever may te the value of the index $k = 1, 2, \ldots, s$.

By indicating

$$\eta_{ij} = a_{ij}\left(\sqrt[n]{\frac{D}{D'}} - 1\right) + \epsilon_{ij}\sqrt[n]{\frac{D}{D'}}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{12}$$

let us examine the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \sum (a_{ij} + \eta_{ij})x_ix_j. \tag{13}$$

By virtue of (12) the form φ_0 is of determinant D.

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‡ Mathematische Annalen, V. XI, p. 250

By choosing the parameter ϵ sufficiently small, one can suppose that

$$|\eta_{ij}| < \eta, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{14}$$

η being a positive parameter as small as one would like.

By virtue of (5), (11) and (12), one obtains

$$\sum \eta_{ij} l_{ik} l_{jk} \geq 0. \quad (k = 1, 2, \ldots, s) \tag{15}$$

By developping the determinant D of the form (13) into series, one will find

$$D + \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} + R_2 = D. \tag{16}$$

In this equality the remainder R_2 verifies an inequality

$$|R_2| < \eta^2 P,$$

P being a positive number not depending on the parameter η so long as $\eta < 1$.

By virtue of (16), one obtains

$$\left| \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} \right| < \eta^2 P. \tag{17}$$

We have suppose that the quadratic form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjointed to the form φ, be interior to the domain R. On the ground of that which has been said in Number 13, one will determine the form $\sum \frac{\partial D}{\partial a_{ij}} \cdot x_i x_j$ with the help of the equality

$$\sum \frac{\partial D}{\partial a_{ij}} x_i x_j = \sum_{k=1}^{s} \rho_k \lambda_k^2 \tag{18}$$

where

$$\rho_k > 0. \quad (k = 1, 2, \ldots, s) \tag{19}$$

The equality (18) can be replaced by the following ones:

$$\frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k l_{ik} l_{jk}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By multiplying these equations by η_{ij} and by adding up the equalities obtained, one will have

$$\sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k \sum \eta_{ij} l_{ik} l_{jk}. \tag{20}$$

By virtue of (15), (17) and (19), one obtains the inequalities

$$0 \leq \sum \eta_{ij} l_{ik} l_{jk} < \eta^2 \frac{P}{\rho_k}; \quad (k = 1, 2, \ldots, s)$$

therefore one can admit

$$\sum \eta_{ij} l_{ik} l_{jk} = \tau_k \eta^2, \quad (k = 1, 2, \ldots, s) \tag{21}$$

and the positive numbers or zeros τ_k $(k = 1, 2, \ldots, s)$ will not exceed fixed limits which do not depend on the parameter η.

After the definition of perfect forms, the equations (21) admit only a single system of solutions. By effecting this solution of equations (21), one obtains

$$\eta_{ij} = \tau_{ij} \eta^2 \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where

$$|\tau_{ij}| < T, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

T being a positive number which does not depend on the parameter η; therefore one will have the inequalities

$$|\eta_{ij}| < \eta^2 T. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{22}$$

This stated, let us take any one positive fraction ϑ and declare

$$\eta = \frac{\vartheta}{T}.$$

By virtue of (14), one will have

$$|\eta_{ij}| < \frac{\vartheta}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and because of (22), it will become

$$|\eta_{ij}| < \frac{\vartheta^2}{T}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By admitting

$$\eta_{ij} < \frac{\vartheta^2}{T},$$

one will have, because of (22),

$$|\eta_{ij}| < \frac{\vartheta^4}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and so on.

One will obtain in this manner the inequalities

$$|\eta_{ij}| < \frac{\vartheta^{2^k}}{T} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; k = 0, 1, 2, \ldots)$$

it follows that

$$\eta_{ij} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By virtue of (12), one obtains

$$\epsilon_{ij} = a_{ij}\left(\sqrt[n]{\frac{D'}{D}} - 1\right); \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

therefore the coefficients ϵ_{ij} are proportional to the coefficients a_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$, which is contrary to the hypothesis.

Properties of the set of domains corresponding to the various perfect forms in n variables.

19 Any perfect form φ will be transformed by an equivalent perfect form φ' with the help of any substitution S of integer coefficients and of determinant ±1.

Let us indicate by R and R' the domains corresponding to the perfect forms φ and φ' and by T the substitution adjointed to the substitution S.

One will easily demonstrate that the domain R will be transformed into an equivalent domain R' with the help of the substitution T.

One concludes that there exists a finite number of domains equivalent to the domain R.

Let us indicate by (R) the set of all the domains corresponding to the various perfect forms in n variables.

The set (R) can be divided into classes of equivalent domains.

On the ground of that which has been previously said, the number of different classes of the set (R) is equal to the number of classes of perfect forms in n variables.

20 *Theorem. Let us suppose that a quadratic form f be interior to a face $P(\mu)$ in μ dimensions of the domain R $(\mu = 1, 2, \ldots, \frac{n(n+1)}{2})$.*

The form f will belong only to the domains of the set (R) which are contiguous through the face $P(\mu)$.

Let us suppose that the domain R be characterised by the quadratic form

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_s^2 \tag{1}$$

and that the face $P(\mu)$ in μ dimensions of the domain R be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2. \tag{2}$$

In the case $\mu = \frac{n(n+1)}{2}$, one will admit $t = s$, and the symbol $P\left(\frac{n(n+1)}{2}\right)$ will indicate the domain R.

The quadratic form f being interior to the face $P(\mu)$, one will have the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2 \text{ where } \rho_k > 0. \ (k = 1, 2, \ldots, t) \tag{3}$$

Let us suppose that the same form f belonged to another domain R' of the set (R).

Let us suppose that the domain R' be characterised by the quadratic forms

$$\lambda_1'^2, \lambda_2'^2, \ldots, \lambda_\sigma'^2 \tag{4}$$

and that the form f be interior to the face $P'(\nu)$ of the domain R' characterised by the quadratic forms

$$\lambda_1'^2, \lambda_2'^2, \ldots, \lambda_\tau'^2. \tag{5}$$

One will have, on the ground of the supposition made,

$$f(x_1, x_2, \ldots, x_n) = \sum_{h=1}^{\tau} \rho_h' \lambda_h'^2 \text{ where } \rho_h' > 0. \ (h = 1, 2, \ldots, \tau) \tag{6}$$

This declared, let us indicate by φ and φ' the perfect forms corresponding to the domains R and R' and suppose, for more simplicity, that the minimum of forms φ and φ' be M.

By indicating with the symbol (f, f') the result

$$(f, f') = \sum a_{ij} a'_{ij},$$

from two quadratic forms

$$f(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j \quad \text{and} \quad f'(x_1, x_2, \ldots, x_n) = \sum a'_{ij} x_i x_j,$$

let us examine two results (f, φ) and (f, φ').

By virtue of (13), one obtains

$$(f, \varphi) = \sum_{k=1}^{t} \rho_k(\varphi, \lambda_k^2) \quad \text{and} \quad (f, \varphi') = \sum_{k=1}^{t} \rho_k(\varphi', \lambda_k^2). \tag{7}$$

By virtue of (6), one obtains

$$(f, \varphi) = \sum_{h=1}^{\tau} \rho'_h(\varphi, {\lambda'_h}^2) \quad \text{and} \quad (f, \varphi') = \sum_{h=1}^{\tau} \rho'_h(\varphi', {\lambda'_h}^2). \tag{8}$$

Let us notice that

$$(\varphi, \lambda_k^2) = M \quad \text{and} \quad (\varphi', \lambda_k^2) \geq M; \quad (k = 1, 2, \ldots, s) \tag{9}$$

$$(\varphi, {\lambda'_h}^2) \geq M \quad \text{and} \quad (\varphi', {\lambda'_h}^2) = M. \quad (h = 1, 2, \ldots, \sigma) \tag{10}$$

From equalities (7), one derives

$$(f, \varphi') - (f, \varphi) = \sum_{k=1}^{t} \rho_k \left[(\varphi', \lambda_k^2) - (\varphi, \lambda_k^2)\right], \tag{11}$$

and by virtue of (3) and (9) there comes

$$(f, \varphi') - (f, \varphi) \geq 0.$$

From equalities (8), one derives

$$(f, \varphi') - (f, \varphi) = \sum_{h=1}^{\tau} \rho'_h \left[(\varphi', {\lambda'_h}^2) - (\varphi, {\lambda'_h}^2)\right], \tag{12}$$

and by virtue of (6) and (10), one will have

$$(f, \varphi') - (f, \varphi) \leq 0.$$

It follows that

$$(f, \varphi') = (f, \varphi),$$

and the equalities (11) and (12) give

$$(\varphi', \lambda_k^2) = (\varphi, \lambda_k^2), \quad (k = 1, 2, \ldots, t)$$

$$(\varphi', {\lambda'_h}^2) = (\varphi, {\lambda'_h}^2). \quad (h = 1, 2, \ldots, \tau)$$

By virtue of (9) and (10), there arrive

$$(\varphi, {\lambda'_h}^2) = M, \quad (h = 1, 2, \ldots, \tau) \tag{13}$$

$$(\varphi', \lambda_k^2) = M. \quad (k = 1, 2, \ldots, t) \tag{14}$$

By virtue of equalities (13), the quadratic forms (5) are found among those of the series (1). By virtue of (14), the quadratic forms (2) are found among those of the series (4).

I argue that in this case the series (2) and (5) contain the same forms.

To demonstrate this, let us suppose that all the forms belonging to the face $P(\mu)$ verify the equations

$$\Psi_1(f) = 0, \ \Psi_2(f) = 0, \ldots, \ \Psi_r(f) = 0$$

and that any form belonging to the domain R verifies the inequalities

$$\Psi_1(f) \geq 0, \ \Psi_2(f) \geq 0, \ldots, \ \Psi_r(f) \geq 0. \tag{15}$$

By virtue of (6), one will have

$$\rho'_1 \Psi_i({\lambda'_1}^2) + \rho'_2 \Psi_i({\lambda'_2}^2) + \ldots + \rho'_\tau({\lambda'_\tau}^2) = 0, \quad (i = 1, 2, \ldots, r)$$

and because of (15), one finds

$$\Psi_i({\lambda'_h}^2) = 0; \quad (i == 1, 2, \ldots, r; h = 1, 2, \ldots, \tau)$$

therefore all the forms of the series (5) belong to the series (2).

In the same way, one will demonstrate that all the forms of the series (2) belong to the series (5).

One concludes that the faces $P(\mu)$ and $P'(\nu)$ coincide, therefore the domains R and R' are contiguous through the face $P(\mu)$.

Corollary. A quadratic form which is interior to a domain of the set (*R*) *can not belong to any other domain of that set.*

21 *Theorem. Let us suppose that to a face $P(\mu)$ of the domain R belong positive quadratic forms. In this case, the number of domains of the set (R) contiguous through the face $P(\mu)$ is finite.*

Let us indicate by

$$R, R_1, R_2, \ldots$$

the domains of the set (R) contiguous through the face $P(\mu)$. Let

$$\varphi, \varphi_1, \varphi_2, \ldots$$

be the corresponding perfect forms having the minimum M.

On the ground of the supposition made, one positive quadratic form f will be interior to the face $P(\mu)$.

We have demonstrated in the previous number that

$$(f, \varphi) = (f, \varphi_1) = (f, \varphi_2) = \ldots \tag{16}$$

It is easy to demonstrate that the number of perfect forms having the minimum M and verifying the equalities (16) is finite.

Algorithm for the search for the domain of the set (R) contiguous to another domain by a face in $\frac{n(n+1)}{2} - 1$ dimensions

22 Let

$$\varphi(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$$

be a perfect form having the minimum M the various representation of which make up a series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \tag{1}$$

Let us suppose that a face P of the domain R corresponding to the perfect form φ be determined by the equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0$$

and by the condition

$$\Psi(f) \geq 0$$

which is verified by any quadratic form belonging to the domain R.

Let us suppose that the face P be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2 \tag{2}$$

where

$$\lambda_k = l_{1k} x_1 + l_{2k} x_2 + \ldots + l_{nk} x_n. \quad (k = 1, 2, \ldots, s)$$

On the ground of the supposition made, one will have the equalities

$$\sum p_{ij} l_{ik} l_{jk} = 0 \;\; (k = 1, 2, \ldots, t) \tag{3}$$

which define the coefficients P_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ to an immediate common factor.

Let us suppose that the face P could belong to the other domains of the set (R). Let us indicate by R' a similar domain. Let φ' be the perfect form corresponding to the domain R'.

By virtue of the supposition made, the quadratic form (2) belong to the domains R and R'. It results in that the systems

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, [\ldots,] l_{n2}), \ldots, (l_{1t}, l_{2t}, \ldots, l_{nt}) \tag{4}$$

corresponding to the forms (2) represent the minimum of forms φ and φ'.

Let us suppose, for more simplicity, that the forms φ and φ' had the minimum M. One will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M \text{ and } \sum a'_{ij} l_{ik} l_{jk} = M, \quad (k = 1, 2, \ldots, t) \tag{5}$$

by putting

$$\varphi'(x_1, x_2, \ldots, x_n) = \sum a'_{ij} x_i x_j.$$

From equation (5), one gets

$$\sum (a'_{ij} - a_{ij}) l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, t)$$

and by virtue of (3), it becomes

$$a'_{ij} = a_{ij} + \rho p_{ij}. \;\; (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{6}$$

Let us indicate

$$\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j.$$

By virtue of (16), one obtains

$$\varphi'(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho \Psi(x_1, x_2, \ldots, x_n). \tag{7}$$

This stated, let us choose in the series (1) a system $(l_{1h}, l_{2h}, \ldots, l_{nh})$ which does not belong to the series (4). As

$$\varphi(l_{1h}, l_{2h}, \ldots, l_{nh}) = M \quad \varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) \geq M$$

and

$$\Psi(l_{1h}, l_{2h}, \ldots, l_{nh}) > 0,$$

one deduces from the equality

$$\varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) = \varphi(l_{1h}, l_{2h}, \ldots, l_{nh}) + \rho\Psi(l_{1h}, l_{2h}, \ldots, l_{nh})$$

the inequality

$$\rho \geq 0.$$

The supposition $\rho = 0$ being obviously impossible, one obtains

$$\rho > 0,$$

and it follows that

$$\varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) > M.$$

Let us indicate by

$$(l'_1, l'_2, \ldots, l'_n), (l''_1, l''_2, \ldots, l''_n), \ldots, (l_1^{(r)}, l_2^{(r)}, \ldots, l_n^{(r)}) \tag{8}$$

all the representations of the minimum of the perfect form φ' which are not found in the series (4). By virtue of (7), one will have

$$\varphi'(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) = \varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) + \rho\Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) \qquad (\mathrm{k} = 1, 2, \ldots, \mathrm{r})$$

which results in

$$\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) > M \text{ and } \Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) < 0. \qquad (k = 1, 2, \ldots, r) \tag{9}$$

The value of the parameter ρ will have for expression

$$\rho = \frac{\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) - 1}{-\Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)})}. \quad (k = 1, 2, \ldots, r)$$

Let us examine any one value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} \tag{10}$$

determined with the condition

$$\Psi(x_1, x_2, \ldots, x_n) < 0. \tag{11}$$

I argue that one will have the inequality

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} \geq \rho.$$

Let us suppose the contrary. By supposing that

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} < \rho,$$

one will find, because of (11),

$$\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) < M$$

or, that which comes to the same thing because of (7),

$$\varphi'(x_1, x_2, \ldots, x_n) < M,$$

which is contrary to the hypothesis.

We have arrived at the following important result:

There exists only a single domain R' contiguous to the domain R through the face P. The corresponding perfect form φ' will be determined by the equality (7) provided that the parameter ρ presents the smallest positive value of the function (10).

Let us notice that by virtue of (3) and (9), all the quadratic forms belonging to the domain R' verify the inequality

$$\Psi(f) \leq 0.$$

One concludes that the domains R and R' are found from two opposite sides of the plane in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation

$$\Psi(f) = 0.$$

23 The smallest positive value of the function (10) can be obtained with the help of operations the number of which is finite.

The whole problem is reduced to the preliminary study of a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequality

$$\Psi(l_1, l_2, \ldots, l_n) < 0$$

and satisfying the condition that the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_0 \Psi(x_1, x_2, \ldots, x_n),$$

where one has admitted

$$\rho_0 \frac{\varphi(l_1, l_2, \ldots, l_n) - M}{-\Psi(l_1, l_2, \ldots, l_n)}$$

be positive.

One will determine in this case all the systems $(x_1, x_2, \ldots, x_n)$ of integers verifying the inequality

$$\varphi(x_1, x_2, \ldots, x_n) \geq M \tag{12}$$

the number of which is finite, and one will find among these systems all those which define the smallest value of the function (10).

Let us indicate by R, as we have done in Number 2, the upper limit of values of the parameter ρ.

The problem is reduced to the study of a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequality

$$\varphi(l_1, l_2, \ldots, l_n) + R\Psi(l_1, l_2, \ldots, l_n) < M. \tag{13}$$

It can turn out that the equation

$$\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n) = 0$$

will be verified by integers, one will determine them with the help of equations

$$\frac{\partial \varphi}{\partial x_i} + R \frac{\partial \Psi}{\partial x_i} = 0. \quad (i = 1, 2, \ldots, n)$$

In the case where these equations can not be verified by any one system of integers, one will study the values of linear forms

$$\frac{\partial \varphi}{\partial x_i} + R \frac{\partial \Psi}{\partial x_i} \quad (i = 1, 2, \ldots, n)$$

and one will determine as many as one wish of the systems of integers verifying the inequality (13).

By supposing that a system of integers $(l_1, l_2, \ldots, l_n)$ verifying the inequality (13) were determined, one can look for the smallest positive value ρ of the function (10) with the help of the following procedure.

The inequality (12) can be put under the form

$$\varphi(x_1, x_2, \ldots, x_n) \left(1 - \frac{\rho_0}{R}\right) + \frac{\rho_0}{R} \left[\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n)\right] \leq M,$$

and as

$$\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n) \geq 0,$$

it becomes

$$\varphi(x_1, x_2, \ldots, x_n) \left(1 - \frac{\rho_0}{R}\right) \leq M,$$

or differently

$$\varphi(x_1, x_2, \ldots, x_n) \leq M \frac{R}{R - \rho_0}.$$

Among the systems of integers verifying this inequality one will find all the systems (8) searched for.

Algorithm for the search for the domain of the set (R) to which belongs an arbitrary positive quadratic form.

24 *Theorem. Any positive quadratic form belongs to at least one domain of the set (R).*

Let

$$f(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$$

be any one positive quadratic form.

Let us choose any one domain R from the set (R).

Let us suppose that the form f, did not belong to the domain (R).

In that case all the linear inequalities which defined the domain R will not be verified. Let us suppose that one had the inequality

$$\Psi(f) = \sum p_{ij} a_{ij} < 0. \tag{1}$$

Let us indicate by R_1 the domain contiguous to the domain R through the face in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation

$$\Psi(f) = 0.$$

By indicating with φ and φ_1 the contiguous perfect forms corresponding to the domains R and R_1, one will have, as we have seen in Number 22,

$$\varphi_1(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) \tag{2}$$

where $\rho > 0$ and $\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j$.

Let us examine two results (f, φ) and (f, φ_1). By virtue of (2), one will have

$$(f, \varphi_1) = (f, \varphi) + \rho(f, \Psi),$$

and as, because of (1),

$$(f, \Psi) = \sum p_{ij} a_{ij} < 0,$$

it becomes

$$(f, \varphi) > (f, \varphi_1).$$

Let us suppose that by proceeding in this manner one obtains a series of domains

$$R, R_1, R_2, \ldots. \tag{3}$$

By indicating with

$$\varphi, \varphi_1, \varphi_2, \ldots \tag{4}$$

the series of corresponding perfect forms, one will have the inequalities

$$(f, \varphi) > (f, \varphi_1) > (f, \varphi_2) > \cdots$$

so long as the form f did not belong to the domains (3).

By noticing that all the perfect forms (4) possess the same minimum M, one will easily demonstrate that the number of perfect forms (4) verifying the integrality

$$(f, \varphi) < P$$

is bounded, whatever may be the value of the positive parameter P.

One concludes that the series of domains (3) will necessarily be terminated by a domain R_m to which belonged the form f considered.

Study of a complete system of domains representing the various classes of the set (R).

25

Let R be any one domain of the set (R). Let us suppose that one had determined all the domains

$$R, R_1, R_2, \ldots, R_\sigma \tag{1}$$

contiguous to the domain R through the various faces in $\frac{n(n+1)}{2} - 1$ dimensions, then let us suppose that one had determined all the domains contiguous to the domains (1) and so on.

I say that by proceeding in this manner one will come across any domain of the set (R) arbitrarily chosen.

For example, if one wish to arrive at a domain $R^{(0)}$, one will choose a positive quadratic form f which is interior to the domain $R^{(0)}$ and one will proceed as we have done in Number 24. One will determine this way a series of domains

$$R, R', R'', \ldots, R^{(\mu)}, R^{(0)}$$

which are successively contiguous through faces in $\frac{n(n+1)}{2} - 1$ dimensions.

We have seen in Number 19 that the set (R) can be divided into classes of equivalent domains the number of which is finite.

Let us find a system of domains representing the various classes of the set (R).

By starting from the domain R, we have determined all the domains

$$R_1, R_2, \ldots, R_\sigma$$

contiguous to the domain R. By not considering equivalent domains as being different, let us choose among the domain (1) those which are not one to one equivalent and are not equivalent to the domain R. Let us suppose that one had obtained the series

$$R, R_1, R_2, \ldots, R_{\mu-1} \tag{2}$$

of domains which are not one to one equivalent.

One will study in the same way the domains contiguous to the domains $R, R_1, R_2, \ldots, R_{\mu-1}$ and one will extend the series (2) by adding to it new domains

$$R_\mu, R_{\mu+1}, \ldots, R_{\nu-1}$$

which are not one to one equivalent and are not equivalent to the domains (2).

By proceeding in this way, one will always obtain a series

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{3}$$

which enjoys the following property: the domain belonging to the series (3) are not one to one equivalent and all the domains contiguous to the domains (3) are equivalent to them.

The series (3) obtained presents a complete system of representations of various classes of the set (R).

26

The study of the series (3) can be facilitated particularly by the help of substitutions which transform into itself the domains of the set (R).

Let us suppose that the domain R corresponding to a perfect form φ be determined by the inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By declaring

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine, as we have seen in Number 22, by the equalities

$$\varphi_k = \varphi + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \sigma) \tag{4}$$

σ perfect forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$. One will call them contiguous to the perfect form φ.

Let us indicate by g the group of substitutions which do not change the perfect form φ.

The perfect forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$ being well determined by the perfect form φ, one concludes that all the substitutions of the group g will only permute the forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$.

By not considering as different the forms in proportional coefficients, one can say, by virtue of (4), that the group g will only permute the quadratic forms

$$\Psi_1, \Psi_2, \ldots, \Psi_\sigma. \tag{5}$$

Let us suppose that one had chosen in this series the forms

$$\Psi_1, \Psi_2, \ldots, \Psi_{\mu-1} \tag{6}$$

which enjoyed the following properties: each form of the series (5) will be transformed into a form of the series (6) with the help of a substitution belonging to the group g, the forms (6) can not be transformed one to one with the help of substitutions of the group g.

The perfect forms

$$\varphi_k = \varphi + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \mu - 1)$$

can replace the perfect forms (4), therefore one will determine only the values of parameters $\rho_1, \rho_2, \ldots, \rho_{\mu-1}$.

The corresponding domains

$$R_1, R_2, \ldots, R_{\mu-1}$$

can replace the domains (1).

It can come to pass that among the domains $R, R_1, R_2, \ldots, R_{\mu-1}$ are found equivalent domains, one will recognise this with the help of particular methods.

On a reduction method of positive quadratic forms.

27

Definition. One will call reduced any positive quadratic form belonging to any one domain

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{1}$$

of a complete system of representations of various classes of the set (R).

Let us suppose that one had determined all the substitutions

$$S_1, S_2, \ldots, S_m \tag{2}$$

which transform the domains contiguous with the domains (1) through the faces in $\frac{n(n+1)}{2} - 1$ dimensions into these domains here.

Let f be any one positive quadratic form which is not reduced. One will determine with the help of the algorithm shown in Number 24 a series of domains

$$R, R', R'', \ldots, R^{(h)}$$

successively contiguous. Let us suppose that the domain $R^{(h)}$ be the first one which does not belong to the series (1).

With the help of a substitution S' which is found among those of the series (2), one will transform the domain $R^{(h)}$ into a domain R_k belonging to the series (1).

By transforming the form f with the help of the substitution S' into an equivalent form f', one will determine with the help of the form f' a new series of domains

$$R_k, R_k', \ldots, R_k^{(t)}$$

and so on.

One will determine in this way a series of substitutions

$$S', S'', \ldots, S^{(\lambda)}$$

which, all, are found among those of the series (2) and the product

$$S = S' S'' \cdots S^{(\lambda)}$$

of which presents a substitution S with the help of which the form f will be transformed into a reduced form.

28

Let us suppose now that two reduced forms f and f' be equivalent.

If one of these forms, for example f, is interior to the domain R_k, the form f' will also be interior to it. One concludes that the form f can be transformed into a form f' only with the help of a substitution which transforms the domain R_k into itself.

Let us suppose that the reduced equivalent forms f and f' be interior to the faces in μ dimensions of domains (1).

In this case one will declare supplementary conditions for the reduced forms f and f'. After having determined all the faces in μ dimensions of domains (1), one will choose a complete system of representatives of these various classes. Let us suppose that this system be formed by the faces in μ dimensions

$$P_1(\mu), P_2(\mu), \ldots, P_\nu(\mu). \tag{3}$$

Any positive quadratic form which is interior to a face in μ dimensions of a domain of the set (R) will be equivalent to a form which is interior to the faces (3), one will call it reduced.

Two reduced positive quadratic forms which are interior to the faces (3) will be equivalent only provided that they be interior to the same face and that the substitution which transforms one of them into another one also transforms this face into itself.

We have arrived at the following result:

A reduced quadratic form can be transformed into another reduced form or into itself only with the help of a substitution which transforms into itself a domain or a face of domains belonging to the series (1).

Second Part

Some applications of the general theory to the study of perfect quadratic forms.

On the principal perfect form

29 We will not consider as different the quadratic forms of proportional coefficients, therefore one can arbitrarily choose the minimum value of a positive quadratic form.

In that which follows, one will study only the perfect quadratic forms whose minimum is 1. One will indicate by D the determinant of these forms.

Among the various perfect forms, one form

$$\varphi = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n \; {}^{\ddagger}$$

where

$$a_{ii} = 1,\ (i = 1,2,\ldots,n),\ a_{ij} = \frac{1}{2},\ (i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j) \text{ and } D = \tfrac{n+1}{2^n}$$

One will call principal the perfect form φ.

The perfect form φ possesses $\frac{n(n+1)}{2}$ representations of the minimum 1, which define $\frac{n(n+1)}{2}$ linear forms

$$\lambda_1 = x_1,\ \lambda_2 = x_2, \ldots,\ \lambda_n = x_n,\ \lambda_{n+1} = x_1 - x_2,\ \lambda_{n+2} = x_1 - x_3, \ldots, \lambda_{\frac{n(n+1)}{2}} = x_{n-1} - x_n.$$

The domain R corresponding to the perfect form φ is made up of all the quadratic forms determined by the equality

$$\sum a_{ij}x_ix_j = \sum_{k=1}^{\frac{n(n+1)}{2}} \rho_k\lambda_k^2 \text{ where } \rho_k \geq 0. \quad (k = 1,2,\ldots,\frac{n(n+1)}{2})$$

From this equality one obtains

$$\rho_k = a_{1k} + a_{2k} + \ldots + a_{nk} \text{ so long as } k = 1,2,\ldots,n,$$

$$\rho_k = -a_{ij} \text{ so long as } k > n;\ (i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j)$$

therefore the domain R will be determined by the following inequalities:

$$\begin{cases} a_{1k} + a_{2k} + \ldots + a_{nk} \geq 0, & (k = 1,2,\ldots,n) \\ -a_{ij} \geq 0. & (i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j) \end{cases} \tag{1}$$

By virtue of (1), the perfect form φ possesses $\frac{n(n+1)}{2}$ contiguous perfect forms which are determined by the equalities

$$\begin{cases} \varphi_k = \varphi + \rho_k x_k(x_1, x_2, \ldots, x_n), & (k = 1,2,\ldots,n) \\ \varphi_k = \varphi - \rho_k x_i x_j, & \left(k = n+1, n+2, \ldots, \frac{n(n+1)}{2}, i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j\right) \end{cases} \tag{2}$$

30 Let us find equivalent forms among the perfect forms contiguous to the perfect form φ.

To this effect, let us determine the group g of substitutions which do not change the form φ.

Let us examine, in the first place, the form adjointed to the form φ.

One will easily demonstrate that the coefficients of the form adjointed to φ are proportional to those of the form

$$\omega = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{\frac{n(n+1)}{2}}^2. \tag{3}$$

One concludes, by virtue of the theorem of Number 17, that the principal perfect form φ is extreme.

The quadratic form ω will have for expression

$$\omega = nx_1^2 + nx_2^2 + \ldots + nx_n^2 - 2x_1x_2 - 2x_1x_3 - \ldots - 2x_{n-1}x_n$$

where

$$a_{ii} = n,\ (i = 1,2,\ldots,n) \quad a_{ij} = -1. (i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j)$$

Let us find all the representations of the minimum of the form ω.

The linear forms

$$x_1, x_2, \ldots, x_n, x_1 + x_2 + \ldots + x_n \tag{4}$$

characterise $n+1$ representations of the value n of the form ω.

I say that the form ω has the minimum n and all the representations of the minimum of the form ω are characterised by the linear form (4).

The form φ has been given for the first time by Zolotareff in a Mémoire titled: On an indeterminate equation of the third degree (in Russian)

To demonstrate this, let us examine any one value $\omega(x_1, x_2, \ldots, x_n)$ of the form ω. By supposing that none of the numbers $x_1, x_2, \ldots, x_n$ becomes zero, one will have by virtue of (13)

$$\omega(x_1, x_2, \ldots, x_n) > n,$$

the system $x_1 = 1,\ x_2 = 1, \ldots, x_n = 1$ being excluded.

Let us suppose that any one of the numbers $x_1, x_2, \ldots, x_n$ does not cancel out and that

$$x_{k+1} = 0,\quad x_{k+2} = 0, \ldots,\quad x_n = 0;$$

one obtains, by virtue of (3),

$$\omega(x_1, x_2, \ldots, x_k, 0, \ldots, 0) = (n-k+1)(x_1^2 + x_2^2 + \ldots + x_k^2) + \sum (x_k - x_h)^2,$$

and it follows that

$$\omega(x_1, x_2, \ldots, x_n) \geq k(n-k+1),$$

therefore

$$\omega(x_1, x_2, \ldots, x_n) > n \quad \text{so long as} \quad k \geq 2.$$

This stated, let us indicate by G the group of substitutions which transform into itself the domain R. By virtue of (3), any substitution of the group G does not change the form ω.

The group g being adjointed to the group G, one concludes that each substitution of the group g will only permute te linear forms (4) by changing the sign of a few among them.

By noticing that

$$x_1^2 + x_2^2 + \ldots + x_n^2 + (x_1 + x_2 + \ldots + x_n)^2 = 2\varphi,$$

one concludes that the group g is composed of all the substitutions which permute the forms

$$x_1^2 + x_2^2 + \ldots + x_n^2 + (x_1 + x_2 + \ldots + x_n)^2.$$

Let us indicate

$$x_0 = -x_1 - x_2 - \ldots - x_n \text{ and } x'_0 = -x'_1 - x'_2 - \ldots - x'_n, \tag{5}$$

and let $k_0, k_1, \ldots, k_n$ be any one permutation of numbers $0, 1, 2, \ldots, n$.

By posing

$$x_i = e_i x'_{k_i} \quad \text{where} \quad e_i = \pm 1, \quad (i = 0, 1, 2, \ldots, n) \tag{6}$$

one will have

$$x_0 + x_1 + \ldots + x_n = e_0 x'_{k_0} + e_1 x'_{k_1} + \ldots + e_n x'_{k_n},$$

and as, because of (5),

$$x_0 = x_1 + \ldots + x_n = 0 \quad \text{and} \quad x'_0, x'_1, \ldots, x'_n = 0,$$

it is necessary that

$$e_0 = e_1 = \ldots = e_n;$$

therefore the equalities (6) reduce to the one here:

$$x_i = e x'_{k_i}. \quad (i = 0, 1, 2, \ldots, n; e = \pm 1) \tag{7}$$

The number of substitutions defined by the formulae obtained is equal to $2 \cdot 1 \cdot 2 \cdots (n+1)$. By not considering as different two substitutions of opposite coefficients, one will say that the group g is composed of $(n_1)!$ different substitutions. §

With the help of substitutions (7), one can transform any perfect form (2) contiguous to the principal form φ into another form contiguous to the form φ, arbitrarily chosen.

We have arrived at the following important result.

All the perfect forms contiguous to the principal perfect form are equivalent.

31 Let us choose one form among those of the series (2). Let us declare

$$\varphi_1 = \varphi - \rho x_1 x_2.$$

All the perfect form contiguous to the form φ are equivalent to the form φ_1.

Let us find the corresponding value of the parameter ρ.

As we have seen in Number 22, the value searched for of ρ presents the smallest value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - 1}{x_1 x_2} \tag{8}$$

determined with condition

$$x_1 x_2 > 0.$$

One will distinguish in the subsequent studies two cases:

$$1).\ n = 2 \quad \text{and} \quad 2).\ n \geq 3.$$

First case:

By comparing two $n = 2$ binary forms

$$\varphi = x_1^2 + x_2^2 + x_1 x_2 \quad \text{and} \quad \varphi_1 = x_1^2 + x_2^2 + x_1 x_2 - \rho x_1 x_2,$$

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§ See: *Minkowski*, Zur Theorie der positiven quadratischen Formen [On the theory of the positive quadratic forms], (This Journal, V. 101, p. 200) one will notice that by making $\rho = 2$ one obtains the form

$$\varphi_1 = x_1^2 + x_2^2 - x_1 x_2$$

which is evidently equivalent to the perfect form φ, therefore the perfect form φ_1 is that which one has searched for.

Second case:

By making

$$x_1 = 1,\ x_2 = 1,\ x_3 = -1,\ x_4 = 0, \ldots,\ x_n = 0,$$

one obtains a value of the function (8) which is equal to 1.

By making $\rho = 1$, one will present the form φ_1 under the following form:

$$\varphi_1 = \frac{1}{2}\left[(x_1, x_2, \ldots, x_n)^2 + (x_1 - x_2)^2 + x_3^2 + \ldots + x_n^2\right]. \tag{9}$$

It results in that the form φ_1 is positive. On the ground of that which has been said in Number 23, one will find now all the systems of integers veifying the inequality

$$\varphi_1(x_1, x_2, \ldots, x_n) \leq 1.$$

By noticing that the inequality

$$\varphi_1(x_1, x_2, \ldots, x_n) < 1$$

is impossible, because the positive form φ_1 has integer values which corresponds to the integer values of variables, one concludes that the form φ_1 is perfect.

With the help of the equality (9), one will easily determine all the presentations of the minimum of the perfect form φ_1.

On the binary and ternary perfect forms and on the domains which correspond to them.

32 The binary principal perfect form

$$\varphi = x_2 + xy + y_2, \quad D = \frac{3}{4}$$

possesses, as we have seen in Number 29, three contiguous perfect forms which are equivalent to the principal form.

One concludes that all the perfect binary forms constitute only a single class of forms equivalent to the principal form.

The domain $\mathcal{R}$ corresponding to the principal form is made up of binary forms (a, b, c) which are determined by the equality

$$ax^2 + 2bxy + cy^2 = \rho x^2 + \rho' y^2 + \rho''(x - y)^2$$

where

$$\rho \geq 0,\ \ \rho' \geq 0,\ \ \rho'' \geq 0$$

It follows that the domain R is determined by the inequalities

$$\rho = a + b \geq 0,\ \ \rho' = -b \geq 0,\ \ \rho'' = c + b \geq 0.$$

By calling reduced the positive binary forms verifying these inequalities, as we have done in Number 27, one will establish a well known method of reduction, due to Mr. Selling. ¶

It results in that the domain R^0 is determined by the inequalities

$$\rho = c - a \geq 0,\ \ \rho' = a + 2b \geq 0,\ \ \rho'' = -b \geq 0.$$

The inequalities obtained only differ from famous conditions of reduction of positive binary quadratic forms due to Lagrange by the choice of the sign of the coefficient b, that which one can arbitrarily make in the method of Lagrange. ‡

33 Let us examine now the ternary perfect forms.

The principal perfect form

$$\varphi = x_2 + y^2 + z^2 + yz + zx + xy, \quad D = \frac{1}{2}$$

possesses six contiguous perfect forms which, all, are equivalent to the perfect form

$$\varphi_1 = x_2 + y^2 + z^2 + yz + zx$$

which we have found in Number 31.

The substitution

$$x = -x',\ y = y',\ z = -y' - z'$$

transforms the form φ_1 into principal form.

One concludes that all the ternary perfect forms form only a single class.

¶ *Selling.* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

‡ See: *Lagrange.* Recherches d'Arithmétique. [Studies in arithmetic] (Oeuvres de Lagrange published by Serret, V. III, p. 698)

Gauss. Disquisitiones arithmeticae, art. 171. (Werke, V. I.)

Lejeune Dirichlet. Vorlesungen über Zahlentheorie [Letcures on number theory], published by Dedekind, (Braunschweig 1894, §64, p. 155)

The domain R corresponding to the principal form is made up of all the ternary quadratic forms $\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix}$ which are determined by the equality

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \rho_1' + \rho_2 y^2 + \rho_3 z^2 + \rho_4 (y-z)^2 + \rho_5 (z-x)^2 + \rho_6 (x-y)^2.$$

The domain R is determined by the inequalities

$$\begin{array}{ccc} \rho_1 = a + b' + b'' \geq 0, & \rho_2 = a' + b'' + b \geq 0, & \rho_3 = a'' + b + b' \geq 0, \\ \rho_4 = -b \geq 0, & \rho_5 = -b' \geq 0, & \rho_6 = -b'' \geq 0. \end{array}$$

By calling reduced the positive ternary quadratic forms belonging to the domain R, one will establish a method of reduction due to Mr. Selling.

The domain R can be partitioned into 24 equivalent parts which can be transformed one into another with the help of 24 substitution adjoined to those which do not change the principal form.

One of these parts, the domain $\mathcal{R}$, will be composed of all the ternary quadratic forms determined by the equality

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \rho_1 x^2 + \rho_2 y^2 + \rho_3 z^2 + \rho_4 (y-z)^2 + \rho_5 \Psi + \rho_{[6]} \omega$$

where

$$\Psi = x^2 + y^2 + z^2 (y-z)^2 + (z-x)^2, \ \omega = x^2 + y^2 + z^2 (y-z)^2 + (z-x)^2 + (x-y)^2.$$

One will determine the domain $\mathcal{R}$ with the help of inequalities

$$\begin{array}{ccc} \rho_1 = a + 2b' + b'' \geq 0, & \rho_2 = a' + b + b' + b'' \geq 0, & \rho_3 = a'' + b + b' + b'' \geq 0, \\ \rho_4 = -b + b' \geq 0, & \rho_5 = -b' + b'' \geq 0, & \rho_6 = -b'' \geq 0. \end{array}$$

The domain $\mathcal{R}$ enjoys the following properties:

1. Any positive ternary quadratic form is equivalent to at least one form belonging to the domain $\mathcal{R}$.
2. Two ternary quadratic forms which are interior to the domain $\mathcal{R}$ can not be equivalent.

By effecting the transformation of the domain $\mathcal{R}$ with the help of all the substitutions of integer coefficients and of determinant ±1, one will make up the set $(\mathcal{R})$ of domains.

Each domain $\mathcal{R}$ belonging to the set $(\mathcal{R})$ possesses six domain contiguous by faces in 5 dimensions.

The domain $\mathcal{R}$ will be transformed into contiguous domain with the help of the following substitutions:

$$S_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix},$$

$$S_4 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Each substitution of this series transforms into itself a corresponding face in 5 dimensions of the domain $\mathcal{R}$ and permutes two domains of the set $(\mathcal{R})$ which are contiguous through this face.

This results in a method for the search for the substitution which transforms a given form into a form belonging to the domain $\mathcal{R}$. This method is analogous to that which has been shown in Number 27.

By calling reduced any positive ternary quadratic form belonging to the domain $\mathcal{R}$, one will establish a new method of reduction of positive ternary quadratic forms which can be considered as a generalisation of the method of reduction of Lagrange.

34 *On the perfect form* $x_1^2 + x_2^2 + \ldots + x_n^2 + x_1 x_3 + x_1 x_4 + \ldots + x_{n-1} x_n$.

Let us examine the perfect form

$$\varphi_1 = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1 x_3 + x_1 x_4 + \ldots + x_{n-1} x_n$$

obtained in Number 31. One has admitted

$$a_{ii} = 1, \ (i = 1, 2, \ldots, n), \ \ a_{12} = 0, \ \ a_{ij} = \frac{1}{2}. \qquad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

It results in that

$$D = \frac{1}{2^{n-2}}.$$

By supposing that $n \geq 4$, one will have $n^2 - n$ representations of the minimum of the form φ_1 the number of which is greater than $\frac{n(n+1)}{2}$.

These representations of the minimum of the form φ_1 will be characterised by the linear forms

$$\begin{cases} \lambda_1 = x_1, \lambda_2 = x_2, \ldots, \lambda_n = x_n, \lambda_{n+1} = x_1 - x_3, \ldots, \lambda_{\frac{n(n+1)}{2}-1} = x_{n-1} - x_n, \ \lambda_{\frac{n(n+1)}{2}} = x_1 + x_2 - x_3, \ldots, \\ \lambda_{\frac{n(n+1)}{2}+n-3} = x_1 + x_2 - x_n, \lambda_{\frac{n(n+1)}{2}+n-2} = x_1 + x_2 - x_3 - x_4, \ldots, \lambda_{n^2-n} = x_1 + x_2 - x_{n-1} - x_n. \end{cases} \tag{1}$$

The domain R_1 corresponding to the perfect form φ_1 is made up of forms deterined by the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n^2-n} \rho_k \lambda_k^2 \ \text{ where } \ \rho_k \geq 0. \quad (k = 1, 2, \ldots, [n^2 - n])$$

Let us find the linear inequalities which define the domain R_1.
The number of these inequalities is so large in deed for $n = 4$.
One will overcome the difficulties which result by the help of a particular method.

35 Let us find the group g_1 of substitutions which do not change the form φ_1.
To this effect, let us introduce in our studies a quadratic form ω determined by the equality

$$\omega = \frac{2}{n-1}(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{n^2-n}^2).$$

After the reductions, one obtains

$$\omega(x_1, x_2, \ldots, x_n) = nx_1^2 + nx_2^2 + 4x_3^2 + \ldots + 4x_n^2 + 2(n-2)x_1x_2 - 4x_1x_3 - \ldots - 4x_1x_n - 4x_2x_3 - \ldots - 4x_2x_n.$$

One can give in the form ω_2 the following expression:

$$\omega(x_1, x_2, \ldots, x_n) = (x_1 - x_2)^2 + (x_1 + x_2)^2 + (x_1 + x_2 - 2x_3)^2 + \ldots + (x_1 + x_2 - 2x_n)^2.$$

It is easy to demonstrate that the form added to the perfect form φ_1 has coefficients which are proportional to those of the form ω.
It follows that the perfect form φ_1 is extreme.
Let us observe that the linear form

$$x_1 + x_2 + \ldots + x_n,\ x_1 - x_2,\ x_3,\ x_4, \ldots,\ x_n$$

characterise n minimum 4 representations of the form ω. In the case $n \geq 5$, other representations of the minimum of the form ω do not exist; in the case $n = 4$, one obtains 12 representations of the minimum of the form ω.
By noticing that

$$\varphi_1 = \frac{1}{2}\left[(x_1 + x_2 + \ldots + x_n)^2 + (x_1 - x_2)^2 + x_3^2 + \ldots + x_n^2\right],$$

one can say that the group g_1, in the case $n \geq 5$, is composed of all the permutations of the forms

$$(x_1 + x_2 + \ldots + x_n)^2,\ (x_1 - x_2)^2,\ x_3^2, \ldots,\ x_n^2.$$

In the case $n = 4$, one will determine by this method only divisor of the group g_1.
By indicating

$$\begin{array}{llll} u_1 = x_1 + x^2 + \ldots + x_n, & u_2 = x_1 - x_2, & u_3 = x_3, & \ldots, \quad u_n = x_n, \\ u_1' = x_1' + x_2' + \ldots + x_n', & u_2' = x_1' - x_2', & u_3' = x_3', & \ldots, \quad u_n' = x_n' \end{array}$$

let us declare

$$u_i = e_i u_{k_i}', \quad (i = 1, 2, \ldots, n) \tag{2}$$

where $e_i = \pm 1$ $(i = 1, 2, \ldots, n)$ and the indices $k_1, k_2, \ldots, k_n$ present any one permutation of numbers $1, 2, \ldots, n$.
Each system of equalities (2) defines a substitution of the group g_1.
One concludes that the group g_1 is composed of $2^{n-1}n!$ different substitutions, in the case $n \geq 5$.

36 Let us suppose that the domain R_1 be determined by the inequalities

$$\sum p_{ij}^{(k)} a_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By indicating

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine, as we have seen in Number 22, σ perfect forms

$$\varphi_1^{(k)} = \varphi_1 + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \sigma) \tag{3}$$

contiguous to the perfect form φ_1.
All the substitutions of the group g_1 will make only one permutation of forms

$$\Psi_1, \Psi_2, \ldots, \Psi_\sigma. \tag{4}$$

Let us effect the transformation of forms (3) and (4) with the help of the sustitution

$$x_1, x_2, \ldots, x_n = x_1',\ x_1 - x_2 = x_2',\ x_3 = x_3', \ldots,\ x_n = x_n'. \tag{5}$$

The series (4) will be transformed into a series

$$\Psi_1', \Psi_2', \ldots, \Psi_\sigma'.$$

Let us indicate by **g** a group of substitutions

$$x_i' = e_i x_{k_i}'', \quad (i = 1, 2, \ldots, n)$$

where $e = \pm 1$ $(i = 1, 2, \ldots, n)$ and $k_1, k_2, \ldots, k_n$ present a permutation of numbers $1, 2, \ldots, n$. Each substitution of the group **g** makes only one permutation of forms (6), and to a similar substitution corresponds a substitution of the group g_1.

By indicating

$$\Psi'_k(x'_1, x'_2, \ldots, x'_n) = \sum P^{(k)}_{ij} x'_i x'_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine with the help of inequalities

$$\sum P^{(k)}_{ij} a_i a_j \geq 0, \quad (k = 1, 2, \ldots, \sigma) \tag{7}$$

a domain $\mathcal{R}$.

The form φ_1 will be transformed into a form

$$\frac{1}{2}({x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2),$$

with the help of the substitution (5), and any system $(x_1, x_2, \ldots, x_n)$ of integers $x_1, x_2, \ldots, x_n$ will be replaced by a system $(x'_1, x'_2, \ldots, x'_n)$ of number, also integer, $x'_1, x'_2, \ldots, x'_n$ satisfying the condition

$$x'_1 + x'_2 + \ldots + x'_n \equiv 0 (\text{mod } 2). \tag{8}$$

It results in that the linear forms (1) which correspond to the various representations of the minimum of the form φ_1 will be replaced by the forms

$$x'_i + x'_j \text{ and } x'_i - x'_j \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

which characterise the various representations of the minimum 2 of the quadratic form ${x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2$, in the set (X') of all the systems $(x'_1, x'_2, \ldots, x'_n)$ of integers $x'_1, x'_2, \ldots, x'_n$ satisfying the condition (8).

One concludes that the edges of the domain $\mathcal{R}$ will be characterised by the quadratic form

$$(x'_i + x'_j)^2 \quad \text{and} \quad (x'_i - x'_j)^2. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

By virtue of (7), one obtains the inequalities

$$P^{(k)}_{ii} + 2P^{(k)}_{ij} + P^{(k)}_{jj} \geq 0 \text{ and } P^{(k)}_{ii} - 2P^{(k)}_{ij} + P^{(k)}_{jj} \geq 0. \quad (\mathrm{k} = 1, 2, \ldots, \sigma; \mathrm{i} = 1, 2, \ldots, \mathrm{n}; \mathrm{j} = 1, 2, \ldots, \mathrm{n}; \mathrm{i} \neq \mathrm{j}) \tag{9}$$

Let us examine any one form

$$\Psi'(x'_1, x'_2, \ldots, x'_n) = \sum P_{ij} x'_i x'_j \tag{10}$$

belonging to the series (6). By virtue of (9), one will have

$$P_{ii} + 2P_{ij} + P_{jj} \geq 0 \text{ and } P_{ii} - 2P_{ij} + P_{jj} \geq 0. \quad (\mathrm{i} = 1, 2, \ldots, \mathrm{n}; \mathrm{j} = 1, 2, \ldots, \mathrm{n}; \mathrm{i} \neq \mathrm{j}) \tag{11}$$

Among these conditions one will find t quantities which define the coefficients of the form (10) to an immediate common factor. All these equalities will be of the form

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1. \tag{12}$$

Let us suppose that there exists a combination of values of k and h satisfying the conditions

$$P_{kk} + 2P_{kh} + P_{hh} > 0 \text{ and } P_{kk} - 2P_{kh} + P_{hh} > 0. \tag{13}$$

By noticing that the coefficient P_{kh} does not enter the other inequalities (11), one concludes that the coefficient P_{kh} remains undetermined.

For all the coefficients of the form (10) to be determined by the conditions (12) to an immediate common factor, it is necessary, the coefficient P_{kh} being independent of other coefficients, that all the coefficients which remain cancel out.

By virtue of inequalities (13), this supposition is impossible, therefore the inequality (12) has to hold for all the values of indices k and h.

One obtains $\frac{n(n-1)}{2}$ conditions

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1. \quad (\mathrm{k} = 1, 2, \ldots, \mathrm{n}; \mathrm{h} = 1, 2, \ldots, \mathrm{n}; \mathrm{k} \neq \mathrm{h}) \tag{14}$$

which serve to determine the coefficients P_{kh} in functions of coefficients

$$P_{11}, P_{22}, \ldots, P_{nn}. \tag{15}$$

The coefficients $P_{11}, P_{22}, \ldots, P_{nn}$ can not be independent, and will be connected by at least $n-1$ equations of the form (12). Therefore, in at least $n-1$ case, one will have the equations of the form

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0. \tag{16}$$

To make short we will call these equations double.

This stated, let us suppose, in the first place, that there exists at least one coefficient among those of the series (15) which does not enter in the double equations (16). One can suppose, to fix the ideas, that P_{11} be such a coefficient.

The coefficient P_{11} being independent, all the coefficients $P_{22}, \ldots, P_{nn}$ will cancel each other and, by virtue of (14), the coefficients

$$P_{23}, P_{24}, \ldots, P_{n-1,n}$$

will also cancel one another out.

The coefficient P_{11} is used for determining the coefficients $P_{12}, P_{13}, \ldots, P_{1n}$ with the help of equations (14) which take the form

$$P_{11} - 2e_{1k}P_{1k} = 0; \quad (k = 2, 3, \ldots, n)$$

it follows that

$$2P_{1k} = e_{1k}P_{11}. \quad (k = 2, 3, \ldots, n)$$

As, on the ground of the supposition made,

$$P_{11} + 2e_{1k}P_{1k} > 0, \quad (k = 2, 3, \ldots, n)$$

it is necessary that

$$P_{11} > 0,$$

and one can declare

$$P_{11} = 1.$$

The form (10) is determined by the equalities obtained, and one will have

$$\Psi'(x_1', x_2', \ldots, x_n') = {x_1'}^2 + e_{12}x_1'x_2' + \ldots + e_{1n}x_1'x_n'. \tag{17}$$

By replacing the variables

$$e_{12}x_2', \ e_{13}x_3', \ldots, \ e_{1n}x_n'$$

by the variables $x_2', \ldots, x_n'$, one will replace the form (17) by the form

$$\Psi'(x_1', x_2', \ldots, x_n') = x_1'(x_1', x_2', \ldots, x_n').$$

Let us suppose, in the second place, that all the coefficients (15) enter in the double equations (16).

At least one of the coefficients (15) is not zero. Let us suppose that $P_{kk} \neq 0$. Following the hypothesis, the coefficient P_{kk} enters in at least one double equation

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0.$$

It follows that

$$P_{kh} = 0 \text{ and } P_{kk} + P_{hh} = 0,$$

therefore the coefficients P_{kk} and P_{hh} are of opposite signs. Let us suppose, to fix the ideas, that

$$P_{11} = -1. \tag{18}$$

By examining the inequalities

$$P_{11} \pm 2P_{1k} + Pkk \geq 0, \quad (k = 2, 3, \ldots, n)$$

one deduces

$$Pkk > 0. \quad (k = 2, 3, \ldots, n)$$

It results in that the double equation

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0$$

has to be impossible so long as $k \geq 2$ and $h \geq 2$, therefore all the double equations will be of the form

$$P_{11} \pm 2P_{1k} + P_{kk} = 0. \quad (k = 2, 3, \ldots, n)$$

From these equations one gets, by virtue of (18),

$$P_{kk} = 1 \text{ and } P_{1k} = 0. \quad (k = 2, 3, \ldots, n) \tag{19}$$

By substituting the values obtained of coefficients $P_{11}, P_{22}, \ldots, P_{nn}$ in the equations

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1, \quad (k = 2, 3, \ldots, n; h = 2, 3, \ldots, n; k \neq h)$$

one obtains, because of (19),

$$P_{kh} = e_{kh} \quad \text{where} \quad e_{kh} = \pm 1. \ (k = 2, 3, \ldots, n; h = 2, 3, \ldots, n; k \neq h)$$

The form (10) will have for expression

$$\Psi'(x_1', x_2', \ldots, x_n') = -{x_1'}^2 + {x_2'}^2 + {x_3'}^2 + \ldots + {x_n'}^2 + 2e_{23}x_2'x_3' + 2e_{24}x_2'x_4' + \ldots + 2e_{n-1,n}x_{n-1}'x_n' \tag{20}$$

where

$$e_{23} = \pm 1, \ e_{24} = \pm 1, \ldots, \ e_{n-1,n} = \pm 1.$$

One obtains in this way $2^{\frac{(n-1)(n-2)}{2}}$ different forms. By permuting the variables and by changing their signs, one will particularly decrease the number of various forms determined by the formula (20).

37 With the help of results obtained, one can easily recognise whether a given quadratic form $\sum a_{ij}x_ix_j$ belongs to the domain $\mathcal{R}$ or not.

One will examine, in the first place, the sums

$$e_{1k}a_{1k} + e_{2k}a_{2k} + \ldots + e_{nk}a_{nk} \text{ where } e_{1k} = \pm 1, e_{2k} = \pm 1, \ldots, e_{nk} = \pm 1 \text{ and } e_{kk} = 1. \quad (k = 1, 2, \ldots, n)$$

All these sums have to be positive or zero. The inequalities

$$a_{kk} - |a_{1k}| - \ldots - |a_{k-1,k}| - |a_{k+1,k}| - \ldots - |a_{nk}| \geq 0, \quad (k = 1, 2, \ldots, n) \tag{21}$$

present the conditions necessary and sufficient for the inequalities

$$e_{1k}a_{1k} + e_{2k}a_{2k} + \ldots + e_{nk}a_{nk} \geq 0 \quad (k = 1, 2, \ldots, n)$$

to hold.

Let us examine, in the second place, the inequalities

$$-a_{11} + a_{22} + a_{33} + \ldots + a_{nn} + 2e_{23}a_{23} + 2e_{24}a_{24} + \ldots + 2e_{n-1,n}a_{n-1,n} \geq 0$$

where

$$e_{23} = \pm 1,\ \ e_{24} = \pm 1, \ldots,\ \ e_{n-1,n} = \pm 1,$$

These inequalities can be replaced by a single one

$$-a_{11} + a_{22} + a_{33} + \ldots + a_{nn} - 2|a_{23}| - 2|a_{24}| - \ldots - 2|a_{n-1,n}| \geq 0.$$

One will present this inequality under the form

$$a_{11} + a_{22} + \ldots + a_{nn} - 2|a_{12}| - 2|a_{13}| - \ldots - 2|a_{n-1,n}| \geq 2\left(a_{11} - |a_{12}| - \ldots - |a_{1n}|\right).$$

By permuting the variables, one obtains n inequalities

$$a_{11}+a_{22}+\ldots+a_{nn}-2|a_{12}|-2|a_{13}|-\ldots-2|a_{n-1,n}| \geq 2\left(a_{kk} - |a_{1k}| - \ldots - |a_{nk}|\right). \text{where } k = 1, 2, \ldots, n \quad (22)$$

We have arrived at the following result. One can easily recognise whether a given positive quadratic form f belong to the domain R_1 or not. To this effect, one will transform the form f by a form f' with the help of the substitution adjointed to the substitution (5) and one will examine $2n$ inequalities (21) and (22). For the form f to belong to the domain R_1, it is necessary and sufficient that the form f' verifies $2n$ inequalities (21) and (22).

38

Let us return now to the perfect forms (3) contiguous to the perfect form φ_1. We have seen that these forms will be transformed with the help of the substitution (5) into forms

$$\frac{1}{2}\left({x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2\right) + \rho_k \Psi'_k(x'_1, x'_2, \ldots, x'_n). \quad (k = 1, 2, \ldots, \sigma)$$

The forms $\Psi'_1, \Psi'_2, \ldots, \Psi'_\sigma$ can be transformed with the help of substitutions belonging to the group **g** into forms

$$\left\{\begin{array}{l} 1).\ \ x'_3(-x'_1 - x'_2 + x'_3 + x'_4 + \ldots + x'_n), \\ 2).\ \ -{x'_2}^2 + {x'_1}^2 + {x'_3}^2 + \ldots + {x'_n}^2 - 2x'_1x'_3 - \ldots - 2x'_1x'_n + 2e_{34}x'_3x'_4 + \ldots + 2e_{n-1,n}x'_{n-1}x'_n, \end{array}\right. \quad (23)$$

where

$$e_{34} = \pm 1, \ldots,\ \ e_{n-1,n} = \pm 1.$$

The inverse substitution to the substitution (5):

$$x'_1 = x_1 + x_2 + \ldots + x_n,\ \ x'_2 = x_1 - x_2,\ \ x'_3 - x_3, \ldots,\ \ x'_n = x_n$$

will transform the forms (23) into forms

$$1).-2x_1x_3$$

$$2).4(x_1x_2 - \delta_{34}x_3x_4 - \ldots - \delta_{n-1,n}x_{n-1}x_n), \text{ where; } \delta_{34} = 0 \text{ or } 1, \ldots, \delta_{n-1,n} = 0 \text{ or } 1.$$

One concludes that all the perfect forms contiguous to the form φ_1 are equivalent to the following perfect forms

$$1).\varphi_1 - \rho x_1x_3,$$

$$2).\varphi_1 + \rho(x_1x_2 - \delta_{34}x_3x_4 - \ldots - \delta_{n-1,n}x_{n-1}x_n),$$

where

$$\delta_{34} = 0 \text{ or } 1, \ldots,\ \ \delta_{n-1,n} = 0 \text{ or } 1.$$

Study of the perfect form $\varphi_1 - \rho x_1x_3$.

39

The perfect form φ_1, possesses, as we have seen in Number 38, many contiguous perfect forms which are not equivalent.

One will determine in the following only a single perfect form

$$\varphi_2 = \varphi_1 - \rho x_1x_3$$

contiguous to the perfect form φ_1.

We have demonstrated in Number 22 that the parameter ρ presents the smallest value of the function

$$\rho = \varphi_1 \frac{(x_1, x_2, \ldots, x_n) - 1}{x_1x_3} \quad (1)$$

determined on condition that

$$x_1x_3 > 0. \quad (2)$$

By declaring

$$x_1 = 1,\ x_2 = 0,\ x_3 = 1,\ x_4 = -1,\ x_5 = 0, \ldots,\ x_n = 0,$$

one obtains the value of the function (1) which is equal to 1, therefore

$$0 < \rho \leq 1. \quad (3)$$

Let us effect the transformation of the function (1) with the help of the substitution

$$x_3 = x'_1,\ -x_1 + x_2 = x'_2,\ x_1 + x_2 + \ldots + x_n = -x'_3,\ x_4 = x'_4, \ldots,\ x_n = x'_n \quad (4)$$

one will have

$$\rho = \frac{{x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2 - 2}{-x'_1(x'_1 + x'_2 + \ldots + x'_n)} \tag{5}$$

where, because of (2),

$$x'_1(x'_1 + x'_2 + \ldots + x'_n) < 0 \tag{6}$$

and, because of (4),

$$x'_1 + x'_2 + \ldots + x'_n \equiv 0 (\mathrm{mod}\ 2),$$

the variables $x'_1 + x'_2 + \ldots + x'_n$ being integers.

Let us indicate

$$f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 + \rho x_1(x_1 + x_2 + \ldots + x_n).$$

By virtue of (5) and (6) the value looked for of ρ is defined by the conditions that the inequality

$$f(x_1, x_2, \ldots, x_n) < 2$$

is impossible, so long as the integers $x_1, x_2, \ldots, x_n$ verify the congruence

$$x_1 + x_2 + \ldots + x_n \equiv 0 (\mathrm{mod}\ 2), \tag{7}$$

and that there exists at least one system $(l_1, l_2, \ldots, l_n)$ verifying the equation

$$f(x_1, x_2, \ldots, x_n) = 2 \tag{8}$$

and the congruence (7).

The form f can be determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \left(x_2 + \rho\frac{x_1}{2}\right)^2 + \left(x_3 + \rho\frac{x_1}{2}\right)^2 + \ldots + \left(x_n + \rho\frac{x_1}{2}\right)^2 + \left(1 + \rho - \frac{n-1}{4}\rho^2\right) x_1^2. \tag{9}$$

It follows that the form f will be positive, provided that

$$1 + \rho - \frac{n-1}{4}\rho^2 > 0,$$

and the upper limit R of values of ρ verifies the equation

$$1 + R - \frac{n-1}{4}R^2 = 1,$$

therefore

$$\frac{R = 2}{\sqrt{n} - 1.} \tag{10}$$

40 This presented, let us examine a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the equation (8) and the congruence (7).

I say that there will be the inequalities

$$\left| l_i + \rho\frac{l_1}{2} \right| \leq 1. \quad (i = 2, \ldots, n) \tag{11}$$

In effect, if one suppose that

$$\left| l_k + \rho\frac{l_1}{2} \right| < 1,$$

one will determine $e_k = \pm 1$ such that the inequality

$$\left| l_k + 2e_k + \frac{l_1}{2} \right| < \left| l_k + \rho\frac{l_1}{2} \right|$$

holds, and one will present

$$l'_i = l_i \text{ and } l'_k = l_k = 2e_k. \quad (i = 1, 2, \ldots, n; i \neq k)$$

The condition (7) will be satisfied, and one will have, by virtue of (9), the inequality

$$f(l'_1, l'_2, \ldots, l'_n) < 2,$$

which is contrary to the hypothesis.

By examining the inequalities (11) and the form f with the help of the formula (9), one will easily demonstrate that among the system of integers verifying the equation (8) with condition (7) is found at least one system $(l_1, l_2, \ldots, l_n)$ satisfying the conditions

$$f(l_1, l_2, \ldots, l_n) = 2 \tag{12}$$

and

$$l_2 = l_3 + \delta,\ l_3 = l_4 = \ldots = l_n \text{ where } \delta = 0 \text{ or } \pm 1. \tag{13}$$

By virtue of (6), one will have the inequality

$$l_1\left[l_1 + \delta + (n-2)l_3\right] < 0.$$

One can suppose that

$$l_1 < 0, \tag{14}$$

and it follows that

$$l_1 + \delta + (n-2)l_3 > 0,$$

therefore, because of (13) and (14), it is necessary that

$$l_3 > 0.$$

I say that $l_3 = 1$. To demonstrate this, let us effect the transformation of the positive quadratic form $f(x_1, x_2, \ldots, x_n)$ with the help of the substitution

$$x_1 = -x,\ x_2 = y,\ x_3 = x_4 = \ldots = x_n - z; \tag{15}$$

one will obtain a ternary positive form

$$F(x,y,z) = x^2 + y^2 + (n-2)z^2 - \rho x(-x + y + (n-2)z).$$

By virtue of the condition (7), the integers x, y, z verify the congruence

$$x + y + (n-2)z \equiv 0 (\mathrm{mod}\ 2). \tag{16}$$

By indicating

$$u = -l_1, \quad v = l_2, \quad w = l_3,$$

one will have, because of (12), (13) and (15),

$$F(u,v,w) = 2,$$

and the condition (16) will be fulfilled.

The inequality

$$f(x,y,z) < 2$$

is impossible so long as the integers x, y, z verify the congruence (16).

Let us effect the transformation of the form $F(x,y,z)$ with the help of the substitution

$$x = x_{,} + y' + (n-2)z',\ \ y = x' - y',\ \ z = z'. \tag{17}$$

The set of systems (x,y,z) of integers verifying the congruence (16) will be replaced by the set of systems (x', y', z') of arbitrary integers.

Let us indicate by $F'(x', y', z')$ the transformed form. Let D and D' be the determinants of forms $F(x,y,z)$ and $F'(x', y', z')$. By virtue of (17), one will have

$$D' = 4D. \tag{18}$$

Let us notice that the number 2 presents the minimum of the form obtained $F'(x', y', z')$ determined in the set of all the systems (x', y', z') of integers, the system $(0,0,0)$ being excluded.

On the ground of the known theorem § on the limit of the minimum of a ternary positive quadratic form, one will have the inequality

$$2 \leq \sqrt[3]{2D'}.$$

It follows that

$$D' \geq 4,$$

and because of (18), one obtains

$$D \geq 1. \tag{19}$$

This presented, let us observe that the form $F(x,y,z)$ has the following values:

$$F(u,v,w) = 2,\ \ F(1,1,0) = 2,\ \ F(1,-1,0) = 2 + 2\rho.$$

By transforming the form $F(x,y,z)$ with the help of the substitution

$$\begin{pmatrix} u, & 1, & 1 \\ v, & 1, & -1 \\ w, & 0, & 0 \end{pmatrix}, \tag{20}$$

one obtains a form

$$F_0(x', y', z') = ax'^2 + a'y'^2 + a''z'^2 + 2by'z' + 2b'z'x' + 2b''x'y',$$

where

$$a = 2,\ a' = 2,\ a'' = 2 + 2\rho \ \text{ and } \ b = \rho. \tag{21}$$

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§ See: *Gauss.* Werke, V. II, p. 192, Göttingen 1863.

Lejeune-Dirichlet. Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen. (This Journal, V. 40, p. 209)

Hermite. Sur la théorie des formes quadratiques ternaires. [On the theory of ternary quadratic forms] (This Journal, V. 40, p. 173)

The product $a \cdot a' \cdot a''$ in any positive ternary quadratic form $\begin{pmatrix} a, & a', & a'' \\ b, & b' & b'' \end{pmatrix}$ is, as one knows, always greater than the determinant of the form, unless the coefficients b, b', b'' do not simultaneously cancel one another out.

By indicating with D_0 the determinant of the form $F_0(x', y', z')$, one will have, because of (21),

$$D_0 < 4(2 + 2\rho),$$

and as, by virtue of (20),

$$D_0 = 4w^2 D,$$

it becomes

$$w^2 D < 2 + 2\rho.$$

By virtue of (3) and (19), one obtains the inequality

$$w^2 < 4,$$

therefore

$$w = 1.$$

41 By returning to the equalities (13), one obtains

$$l_1 = -u, \ \ l_2 = \delta \ \text{ and } l_3 = 1, \ \ l_4 = 1, \ldots, \ \ l_n = 1,$$

where

$$u > 0 \quad \text{and} \quad \delta = 0, 1, 2.$$

By substituting the values found of $l_1, l_2, \ldots, l_n$ in the function (5), one will have

$$\rho = \frac{u^2 + \delta^2 + n - 4}{u(-u + \delta + n - 2)} \tag{22}$$

It remains to determine the smallest value of this function providing that

$$u > 0, \ -u + \delta + n - 2 > 0, \ u \equiv n + \delta (\text{mod } 2) \ \text{ and } \ \delta = 0, 1, 2. \tag{23}$$

Let us admit

$$u = \sqrt{n} - 1 + \alpha, \tag{24}$$

α being a real number.

The function (22) takes the form

$$\rho = \frac{2n + (2\alpha - 2)\sqrt{n} + \alpha^2 - 2\alpha + \delta^2 - 3}{\sqrt[n]{n} + (\alpha - 2)n + (\delta - 2\alpha)\sqrt{n} + 1 - \alpha^2 + \alpha\delta - \delta}.$$

The value searched for of ρ has to verify the inequality

$$\rho < R,$$

therefore because of (10), one will have

$$\frac{2}{\sqrt{n} - 1} - \rho > 0. \tag{25}$$

After the reductions, one obtains

$$\frac{2}{\sqrt{n} - 1} - \rho = \frac{(1 - \delta^2 + 2\delta - \alpha^2)\sqrt{n} + \delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta}{(\sqrt{n} - 1)\left[n\sqrt{n} + (\alpha - 2)n + (\delta - 2\alpha)\sqrt{n} + 1 - \alpha^2 + \alpha\delta - \delta\right]}$$

and, because of (25), it becomes

$$(1 - \delta^2 + 2\delta - \alpha^2)\sqrt{n} + \delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta > 0.$$

By noticing that

$$\delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta \leq 0 \quad \text{so long as} \quad \delta = 0, 1, 2,$$

one obtains the inequality

$$1 - \delta^2 + 2\delta - \alpha^2 > 0.$$

By making $\delta = 0$ and 2, one will have

$$\alpha^2 < 1 \ \text{ as long as } \ \delta = 0 \ \text{ and } \ 2 \tag{26}$$

By making $\delta = 1$, one will have

$$\alpha^2 < 2 \quad \text{so long as} \quad \delta = 1. \tag{27}$$

Let us indicate by m a positive integer determined with the help of inequalities

$$\sqrt{n} - 1 \leq m < \sqrt{n}. \tag{28}$$

By declaring

$$n = m^2 + p, \tag{29}$$

one will have a positive integer p verifying the inequalities

$$0 < p \leq 2m + 1. \tag{30}$$

First case: p is an odd number.

By virtue of (23) and (29), one will have a congruence

$$u \equiv m^2 + p + \delta (\text{mod } 2),$$

p being an odd number; one can declare

$$u = m^2 + \delta + 1 + 2t. \tag{31}$$

By declaring

$$\sqrt{n} = m + \xi,$$

one will have

$$0 < \xi \leq 1, \tag{32}$$

because of (28). By virtue of (24), one obtains the equality

$$u = m - 1 + \xi + \alpha,$$

and because of (31), it becomes

$$\xi + \alpha = m^2 - m + 2 + 2t + \delta. \tag{33}$$

By supposing that $\delta = 0$ or 2, one obtains

$$\xi + \alpha \equiv 0 (\text{mod } 2).$$

By virtue of (26) and (32), it is necessary that

$$\xi + \alpha = 0,$$

therefore

$$u = m - 1 \text{ so long as } \delta = 0 \text{ and } \delta = 2.$$

By supposing that $\delta = 1$, one obtains, because of (33),

$$\xi + \alpha \equiv 1 (\text{mod } 2).$$

By virtue of (27) and (32), the integer $\xi + \alpha$ can have only two values

$$\xi + \alpha = \pm 1,$$

and it results in that

$$u = m \quad \text{or} \quad m - 2 \quad \text{as long as} \quad \delta = 1.$$

One obtains four values of the function (22):

$$\rho_1 = \frac{(m-1)^2+n-4}{(m-1)(n-m-1)}, \qquad \rho_2 = \frac{m^2+n-3}{m(n-m-1)},$$
$$\rho_3 = \frac{(m-2)^2+n-3}{(m-2)(n-m+1)}, \quad \rho_4 = \frac{(m-2)^2+n}{(m-1)(n-m+1)}$$

among which is found the smallest value looked for of ρ.

By noticing that

$$\rho_1 - \rho_2 = \frac{p-3}{m(m-1)(n-m-1)},$$
$$\rho_4 - \rho_1 = \frac{2p+2}{(m-1)(n-m-1)(n-m+1)},$$
$$\rho_3 - \rho_4 = \frac{p+1}{(m-1)(m-2)(n-m+1)},$$

one obtains, because of (30),

$$\rho_1 < \rho_4 < \rho_3$$

and

$$\rho_2 \leq \rho_1 \quad \text{so long as} \quad p \geq 3,$$
$$\rho_1 < \rho_2 \quad \text{so long as} \quad p < 3.$$

There exists only a single odd value of p verifying the inequalities $0 < p < 3$, therefore one will have the inequality

$$\rho_1 < \rho_2 \quad \text{so long as} \quad p = 1.$$

We have arrived at the following result.

The smallest value of ρ will have for expression

$$\rho = \frac{m^2 + n - 3}{m(n - m - 1),} \tag{34}$$

provided that $n = m^2 + p$, and the odd number p verifies the inequalities

$$3 \leq p \leq 2m + 1.$$

In the case $n = m^2 + 1$, *the smallest value of* ρ *will be*

$$\rho = \frac{(m-1)^2 + n - 4}{(m-1)(n-m-1)}.$$

Second case: p is an even number.

One will have, because of (23), the inequality [sic]

$$u \equiv m^2 + \delta (\mathrm{mod}\ 2).$$

By presenting

$$u = m^2 + \delta + 2t,$$

one will have the equalities

$$u = m - 1 + \xi + \alpha \quad \text{and} \quad \xi + \alpha = m^2 - m + 2t + \delta + 1.$$

By supposing that $\delta = 0$ or 2, one obtains

$$\xi + \alpha = 1,$$

and it follows that

$$u = m \quad \text{so long as} \quad \delta = 0 \quad \text{and} \quad 2.$$

By supposing that $\delta = 1$, one obtains

$$\xi + \alpha = 0 \quad \text{or} \quad \xi + \alpha = 2,$$

therefore

$$u = m + 1 \quad \text{or} \quad u = m + 1 \quad \text{so long as} \quad \delta = 1.$$

The smallest value of ρ is found among the following values of the function (22):

$$\rho_1 = \tfrac{m^2+n-4}{m(n-m-2)}, \quad \rho_2 = \tfrac{(m+1)^2+n-3}{(m+1)(n-m-2)},$$
$$\rho_3 = \tfrac{(m-1)^2+n-3}{(m-1)(n-m)}, \quad \rho_4 = \tfrac{m^2+n}{m(n-m)}.$$

By noticing that

$$\rho_2 - \rho_1 = \frac{2m + 4 - p}{m(m+1)(n-m-2)},$$
$$\rho_1 - \rho_4 = \frac{4m - 2p}{m(n-m)(n-m-2)},$$
$$\rho_4 - \rho_3 = \frac{2m - p}{m(m-1)(n-m)},$$

one obtains, because of (30),

$$\rho_3 \leq \rho_4 \leq \rho_1 < \rho_2.$$

We have arrived at the following result:

The smallest value of ρ *is expressed by the equality*

$$\rho = \frac{(m-1)^2 + n - 3}{(m-1)(n-m)}$$

provided that $n = m^2 + p$, *and the even number* p *verifies the inequalities*

$$0 < p < 2m + 1.$$

42 We have determined the value of the parameter ρ which defines the perfect form $\varphi_1 + \rho x_1 x_3$. The determinant D of this form, by virtue of (4) and (9), will have for expression

$$D = \frac{4 + 4\rho - (n-1)\rho^2}{2^n}. \tag{35}$$

The corresponding value of the function $\mathcal{M}(a_{ij})$ defined in Number 16 will be

$$\mathcal{M}(a_{ij}) = 2 \sqrt[n]{\frac{1}{4 + 4\rho - (n-1)\rho^2}}.$$

By applying the formulae obtained to the case:

$$n = 4, 5, 6, 7, 8,$$

one obtains the same value of ρ

$$\rho = 1.$$

The corresponding perfect forms will be

$$x_1^2 + x_2^2 + \ldots, x_4^2 + x_1x_4 + \ldots + x_3x_4,\ D = \frac{5}{2^4},\ \mathcal{M}(a_{ij}) = 2\ \sqrt[4]{\frac{1}{5}};$$

$$x_1^2 + x_2^2 + \ldots, x_5^2 + x_1x_4 + \ldots + x_4x_5,\ D = \frac{4}{2^5},\ \mathcal{M}(a_{ij}) = 2\ \sqrt[5]{\frac{1}{4}};$$

$$x_1^2 + x_2^2 + \ldots, x_6^2 + x_1x_4 + \ldots + x_5x_6,\ D = \frac{3}{2^6},\ \mathcal{M}(a_{ij}) = 2\ \sqrt[6]{\frac{1}{3}};$$

$$x_1^2 + x_2^2 + \ldots, x_7^2 + x_1x_4 + \ldots + x_6x_7,\ D = \frac{2}{2^7},\ \mathcal{M}(a_{ij}) = 2\ \sqrt[7]{\frac{1}{2}};$$

$$x_1^2 + x_2^2 + \ldots, x_8^2 + x_1x_4 + \ldots + x_7x_8,\ D = \frac{1}{2^8},\ \mathcal{M}(a_{ij}) = 2.$$

One comes across all these perfect forms in the Mémoire of Mr.'s Korkine and Zolotareff: Sur les formes quadratiques. [On the quadratic forms] ‡

The formulae obtained give a mean for the study of various perfect forms which verify the inequality

$$\mathcal{M}(a_{ij}) > 2.$$

By making, for example, $n = 12$, one will have

$$m = 3 \quad \text{and} \quad p = 3.$$

By virtue of (34), one obtains

$$\rho = \frac{3}{4},$$

therefore, because of (35),

$$D = \frac{13}{16} \cdot \frac{1}{2^{12}},$$

and it follows that

$$\mathcal{M}(a_{ij}) = 2\ \sqrt[12]{\frac{16}{13}} > 2.$$

All the extreme forms studied by Mr.'s Korkine and Zolotareff do not give a function $\mathcal{M}(a_{ij})$ of values which exceed 2.

On the quadratic perfect forms and on the domains which correspond to them.

43 We have seen in Number 29 that to the quaternary principal perfect form

$$\varphi = x_1^2 + x_2^2 + x_3^2, x_4^2, x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, \quad D = \frac{5}{2^4}$$

corresponds the domain R made up of forms

$$\rho_1\mathrm{x}_1^2 + \rho_2\mathrm{x}_2^2 + \rho_3\mathrm{x}_3^2 + \rho_4\mathrm{x}_4^2 + \rho_5(\mathrm{x}_1 - \mathrm{x}_2)^2 + \rho_{[6]}(\mathrm{x}_1 - \mathrm{x}_3)^2 + \rho_7(\mathrm{x}_1 - \mathrm{x}_4)^2 + \rho_8(\mathrm{x}_2 - \mathrm{x}_3)^2 + \rho_9(\mathrm{x}_2 - \mathrm{x}_4)^2 + \rho_{10}(\mathrm{x}_3 - \mathrm{x}_4)^2.$$

All the perfect forms contiguous to the principal form φ are equivalent to the form

$$\varphi_1 = x_1^2 + x_2^2, x_3^2, x_4^2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, \quad D = \frac{1}{4}.$$

The corresponding domain R_1 is made up of forms

$$\rho_1x_1^2 + \rho_2x_2^2 + \rho_3x_3^2 + \rho_4x_4^2 + \rho_5(x_1 - x_3)^2 + \rho_6(x_1 - x_4)^2 + \rho_7(x_2 - x_3)^2 + \rho_8(x_2 - x_4)^2 +$$
$$\rho_9(x_3 - x_4)^2 + \rho_{10}(x_1 + x_2 - x_3)^2 + \rho_{11}(x_1 + x_2 - x_4)^2, \rho_{12}(x_1 + x_2 - x_3 - x_4)^2.$$

Let us examine the perfect forms contiguous to the perfect form φ_1.

We have demonstrated in Number 38 that all these forms are equivalent to the forms

$$1).\ \varphi_1 - \rho x_1x_3,$$
$$2).\ \varphi_1 + \rho(x_1x_2 - \delta x_3x_4), \quad \text{where } \delta = 0 \text{ or } 1.$$

Let us examine three perfect forms

$$1).\ \varphi_1 + \rho x_1x_2, \quad 2).\ \varphi_1 - \rho x_1x_3, \quad 3).\ \varphi_1 + \rho(x_1x_2 - x_3x_4).$$

1). By making $\rho = 1$ in the form $\varphi_1 + \rho x_1x_2$, one obtains the principal perfect form φ.

2). Let us notice that the form $\varphi_1 - \rho x_1x_3$ is equivalent to the form $\varphi_1 + \rho x_1x_2$.

In effect, the substitution

$$x_1 = -x'_1,\ x_2 = x'_3,\ x_3 = x'_2,\ x_4 = x'_1 + x'_4$$

does not change the form φ_1 and transforms the form x_1x_2 into the form $-x'_1x'_3$.

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‡ Mathematische Annalen, V. VI, p. 367.

3). By making $\rho = 1$ in the form $\varphi_1 + \rho(x_1x_2 - x_3x_4)$, one obtains the form

$$x_1^2 + x_2^2, x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$$

which is evidently equivalent to the perfect form φ_1.

One concludes that all the perfect forms contiguous to the perfect form φ_1 are equivalent to the forms φ and φ_1.

It follows that the set of all the quaternary perfect forms be divided into two classes represented by the perfect forms φ and φ_1.

The set (R) of domains corresponding to various quaternary perfect forms is made up of two classes, too, represented by the domains R and R_1.

On the perfect forms in five variables and on the domains which correspond to them.

44 We have determined two perfect forms in five variables

$$\varphi = x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_4x_5, \quad D = \frac{6}{2^5},$$

$$\varphi_1 = x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_3 + x_1x_4 + \ldots + x_4x_5, \quad D = \frac{4}{2^5}.$$

The corresponding domains R and R_1 will be composed of forms

$$R)\ \rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_2)^2 + \rho_7(x_1 - x_3)^2 + \ldots + \rho_{15}(x_4 - x_5)^2,$$

$$R_1)\ \rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_3)^2 + \ldots + \rho_{20}(x_1 + x_2 - x_4 - x_5)^2.$$

Examine the perfect forms contiguous to the perfect face φ_1. We have demonstrated in Number 38 that all these forms are equivalent to the forms

$$\begin{aligned} &1).\quad \varphi_1 - \rho x_1x_3, \\ &2).\quad \varphi_1 - \rho(x_1x_2 - \delta x_3x_4 - \delta' x_3x_5 - \delta'' x_4x_5). \end{aligned} \tag{1}$$

where

$$\delta = 0 \text{ or } 1, \quad \delta' = 0 \text{ or } 1, \quad \delta'' = 0 \text{ or } 1.$$

In the second case one obtains 8 perfect forms. By permuting the variables x_3, x_4, x_5 one will replace the forms (1) by 4 forms; thus all the perfect forms contiguous to the perfect form φ_1 are equivalent to the 5 following forms:

$$1).\ \varphi_1 + \rho x_1x_2, \quad 2).\ \varphi_1 + \rho x_1x_3, \quad 3).\ \varphi_1 + \rho(x_1x_2 - x_4x_5),$$

$$4).\ \varphi_1 + \rho(x_1x_2 - x_3x_5 - x_4x_5), \quad 5).\ \varphi_1 + \rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5).$$

1). By making $\rho = 1$ in the perfect form $\varphi_1 + \rho x_1x_2$, one obtains the perfect form φ.

2). We have seen in Number 42 that the perfect form $\varphi_1 - \rho x_1x_3$ is determined by the value $\rho = 1$ of the parameter ρ in the case $n = 5$. One obtains the form

$$\varphi_1' = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_1x_4 + x_1x_5 + \ldots + x_4x_5 \tag{2}$$

which will be transformed with the help of the substitution

$$x_1 = -x_2^2,\ x_2 = x_1' - x_2',\ x_3 = x_3',\ x_4 = x_2 + x_4',\ x_5 = x_2' + x_5'$$

into a perfect form φ_1.

3). In the form $\varphi_1 + \rho(x_1x_2 - x_4x_5)$, one will put $\rho = 0$ and one will obtain the form

$$x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_3x_5$$

which is evidently equivalent to the form φ_1.

4). In the form $\varphi_1 + \rho(x_1x_2 - x_3x_5 - x_4x_5))$, one will put $\rho = 1$ and one will obtain the form

$$x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_2x_5$$

which is evident to the perfect form (2)

5). It remains only to determine the perfect form:

$$\varphi_1 + \rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5). \tag{3}$$

By effecting the transformation with the help of the substitution

$$-x_1 + x_2 = x_1',\ \ x_1 + x_2 + x_3 + x_4 + x_5 = x_2',\ \ x_3 = x_3',\ \ x_4 = x_4',\ \ x_5 = x_5' \tag{4}$$

of the form

$$2\varphi_1 + 2\rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5),$$

one obtains the form

$$\mathrm{x}_1'^{\,2} + \mathrm{x}_2'^{\,2} + \mathrm{x}_3'^{\,2} + \mathrm{x}_4'^{\,2} + \mathrm{x}_5'^{\,2} + \frac{\rho}{2}\left[-\mathrm{x}_1'^{\,2} + \mathrm{x}_2'^{\,2} + \mathrm{x}_3'^{\,2} + \mathrm{x}_4'^{\,2} + \mathrm{x}_5'^{\,2} - 2\mathrm{x}_2'\mathrm{x}_3' - 2\mathrm{x}_2'\mathrm{x}_4' - 2\mathrm{x}_2'\mathrm{x}_5' - 2\mathrm{x}_3'\mathrm{x}_4' - 2\mathrm{x}_3'\mathrm{x}_5' - 2\mathrm{x}_4'\mathrm{x}_5'\right]. \tag{5}$$

By virtue of (4) the integer variables $x_1', x_2', x_3', x_4', x_5'$ verify the congruence

$$x_1' + x_2' + x_3' + x_4' + x_5' \equiv 0(\mathrm{mod} 2). \tag{6}$$

By applying to the form (5) the method unveiled in Number 2, one will determine the value of the upper limit $R > 0$ of value $\frac{\rho}{2}$ with the help of equations

$$\xi_1 - R\xi_1 = 0,\ \xi_2 + R(\xi_2 - \xi_3 - \xi_4 - \xi_5) = 0,\ \xi_3 + R(-\xi_2 + \xi_3 - \xi_4 - \xi_5) = 0,$$
$$\xi_4 + R(-\xi_2 - \xi_3 + \xi_4 - \xi_5) = 0,\ \xi_5 + R(-\xi_2 - \xi_3 - \xi_4 + \xi_5) = 0.$$

It results in that

$$\xi_2 = \xi_3 = \xi_4 = \xi_5,$$

and one obtains the equations

$$\xi_1(1 - R) = 0 \quad \text{and} \quad \xi_2(1 - 2R) = 0,$$

thus

$$R = \frac{1}{2}.$$

By declaring

$$x'_1 = 0,\ x'_2 = 1,\ x'_3 = 1,\ x'_4 = 1, x'_5 = 1, \tag{7}$$

one will satisfy the condition (6) and one will have the value $4 - 4\rho$ of the form (5).

By making

$$4 - 4\rho = 2,$$

one obtains

$$\rho = \frac{1}{2}.$$

It follows that the positive quadratic form

$${x'_1}^2 + {x'_2}^2 + \ldots + {x'_5}^2 + \frac{1}{4}\left[-{x'_1}^2 + {x'_2}^2 + {x'_3}^2 + {x'_4}^2 + {x'_5}^2 - 2x'_2x'_3 - \ldots - 2x'_4x'_5\right] \tag{8}$$

will have a value 2 corresponding to the system (7).

By virtue of that which has been discussed in Number 23, the smallest value of the form (8) will correspond to a system $(l_1, l_2, \ldots, l_5)$ verifying the inequality

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \leq 2 \cdot \frac{R}{R - \frac{1}{4}} \quad \text{where} \quad R = \frac{1}{2}.$$

One obtains the inequality

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \leq 4.$$

It is easy to demonstrate that the system (7) is the only one verifying this inequality on condition (6), the systems which verify the inequality

$$-{x'_1}^2 + {x'_2}^2 + {x'_3}^2 + {x'_4}^2 + {x'_5}^2 - 2x'_2x'_3 - 2x'_2x'_4 - 2x'_2x'_5 - 2x'_3x'_4 - 2x'_3x'_5 - 2x'_4x'_5 \geq 0$$

being excluded. By making $\rho = \frac{1}{2}$ in the form (3), one obtains the perfect form

$$\varphi_2 = x'_1 + x'_2 + \ldots + x'_5 + \frac{1}{2}x_1x_2 + x_1x_3 + \ldots + x_2x_5 + \frac{1}{2}x_3x_4 + \frac{1}{2}x_3x_5 + \frac{1}{2}x_4x_5, \quad D = \left(\frac{3}{2}\right)^4 \frac{1}{2^5}.$$

The corresponding domain R^2 is composed of forms

$$\rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_3)^2 + \ldots + \rho_{11}(x_2 - x_5)^2$$
$$+\rho_{12}(x_1 + x_2 - x_3 - x_4)^2 + \rho_{13}(x_1 + x_2 - x_3 - x_5)^2$$
$$+\rho_{14}(x_1 + x_2 - x_4 - x_5)^2 + \rho_{15}(-x_1 - x_2 + x_3 + x_4 + x_5)^2.$$

The number of parameter $\rho_1, \rho_2, \ldots, \rho_{15}$ being equal to the number of dimensions of the domain R_2, one will determine without trouble 15 inequalities which define the domain R_2.

45

We have demonstrated that all the perfect forms contiguous to the perfect form φ_1 are equivalent to the perfect forms φ, φ_1 and φ_2.

Choose the perfect forms contiguous to the perfect form φ_2.

To this effect let us notice, in the first place, that the perfect form φ_1 is contiguous to the perfect form φ_2, then observe that all the perfect forms contiguous to the form φ_2 are equivalent.

To demonstrate this, examine all the faces in 14 dimensions of the domain R_2.

The domain R_2 is characterised by 15 quadratic forms

$$\left\{\begin{array}{l} x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_1 - x_5)^2, (x_2 - x_3)^2, \\ (x_2 - x_4^2), (x_2 - x_5)^2, (x_1 + x_2 - x_3 - x_4)^2, (x_1 + x_2 - x_3 - x_5)^2, \\ (x_1 + x_2 - x_4 - x_5)^2, (-x_1 - x_2 + x_3 + x_4 + x_5)^2. \end{array}\right. \tag{9}$$

Each face in 14 domains of the domain R_2 possesses 14 of these, and the form which remains can be called form opposite to the face.

One concludes that each face is well determined by the opposite face.

For the perfect forms contiguous to the perfect form φ_2 to be equivalent, it is necessary and sufficient that all the faces of the domain R_2 could be transformed one to one with the help of substitutions which do not change the domain R_2.

It would be easy to write all these substitutions, but one will proceed in another way, more speedy.

Let us observe that the face P belonging to the domain R_1 and R_2 is characterised by all the forms (9), the form $(-x_1 - x_2 + x_3 + x_4 + x_5)^2$ being excluded.

With the aid of substitution associated with the substitution (4), one will replace the forms (9) by the forms:

$$\begin{cases} (x_1' \pm x_2')^2, (x_1' \pm x_3')^2, (x_1' \pm x_4')^2, (x_1' \pm x_5')^2, \\ (x_2' + x_3')^2, (x_2' + x_4')^2, (x_2' + x_5')^2, (x_3' + x_4')^2, (x_3' + x_{[}5]')^2, \\ (x_4' + x_5')^2, (x_2' + x_3' + x_4' + x_5')^2. \end{cases} \tag{10}$$

By changing the sign of x_1' and by permuting the variables x_2', x_3', x_4', x_5', one will transform into itself the forms (10), and the form $(x_2' + x_3' + x_4' + x_5')^2$ will not change.

To each similar substitution corresponds a substitution which transforms into itself the domain R_2 and the face P of the domain R_2, and does not change the form $(-x_1 - x_2 + x_3 + x_4 + x_5)^2$.

By changing the sign of x_1' and by permuting x_2', x_3', x_4', x_5', one will transform the form $(x_1' + x_2')^2$ into forms

$$(x_1' \pm x_2')^2, (x_1' \pm x_3')^2, (x_1' \pm x_4')^2, (x_1' \pm x_5')^2$$

and one will transform the form $(x_2' + x_3')^2$ into forms

$$(x_2' + x_3')^2, (x_2' + x_4')^2, (x_2' + x_5')^2, (x_3' + x_4')^2, (x_3' + x_{[5]}')^2, (x_4' + x_5')^2.$$

Thus only the forms

$$(x_1' + x_2')^2,\ (x_2' + x_3')^2,\ (x_2' + x_3' + x_4' + x_5')^2 \tag{11}$$

remain to examine.

By returning to the forms (9), one obtains the forms corresponding to the forms (11).

$$x_2^2, x_3^2, (-x_1 - x_2 + x_3 + x_4 + x_5)^2. \tag{12}$$

It is demonstrated that all the forms (9) can be transformed into forms (12) with the help of substitutions which do not change the domain R_2.

With the help of substitutions

$$x_1 = x_2' - x_5',\ x_2 = x_3',\ x_3 = x_2',\ x_4 = x_1' + x_2',\ x_4 = x_1' + x_2' - x_4' - x_5',\ x_5 = -x_1' + x_3'$$

and

$$x_1 = x_1',\ x_2 = -x_1' - x_2' + x_3' + x_4' + x_5',\ x_3 = x_3',\ x_4 = x_4',\ x_5 = x_5',$$

one will transform the domain R_2 into itself, and the form x_2^2 will be transformed into forms x_3^2 and $(-x_1' - x_2' + x_3' + x_4' + x_5')^2$.

We have demonstrated that all the forms of the domain R_2 are equivalent. It results in, from that we have seen, that all the perfect forms contiguous to the perfect form φ_2 are equivalent to the perfect form φ_1.

One concludes that all the perfect forms in five variables constitute three different classes represented by the perfect forms φ, φ_1 and φ_2.

The set of domains (R) can be divided into three classes also, represented by the domains R, R_1, and R_2.

End of the first Mémoire.

§ D.3 G. F. Voronoi, 1908 (II)

New applications of continuous parameters to the theory of quadratic forms
Second Memoir
Research on the primitive parallelohedron
by Mr. Georges Voronoï in Warsaw
[*Journal für die reine und angewandte Mathematik*, V. 134, 1908]
[translated by K N Tiyapan]

The well known method of reduction for the binary, ternary and quaternary positive quadratic forms † rests upon a property of the positive quadratic form, to know:

Every positive quadratic form $\sum_{i=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ *has n variables in the set E composing all of the systems* $(x_1, x_2, \ldots, x_n)$ *of integers of the variables* x_1, x_2, $\ldots$, x_n *n consecutive minima*

$$M_1 \leq M_2 \leq \cdots \leq M_n$$

determined at condition which the determinant ω *of a system*

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1n}, l_{2n}, \ldots, l_{nn}) \tag{1}$$

which represent these minima in the set E does not vanish.

In all the cases where one has

$$\omega = \pm 1$$

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† *Lagrange,* Recherches d'Arithmétique [Studies in arithmetic] (*Oeuvres,* V. III, p. 695)

Gauß, Disquisitiones arithméticae (*Oeuvres,* V. I, art. 171, p. 146)

Lejeune-Dirichlet, Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen [On the reduction of the positive quadratic forms with three indeterminate integers] (*Oeuvres,* V. II, p. 41)

Minkowski, Sur la réduction des formes quadratiques positives quaternaires [On the reduction of the quaternary positive quadratic forms] (*Comptes Rendus des séances de l'Académie de Paris,* V. 96, p. 1205) one can transform the quadratic form $\sum\sum a_{ij}x_ix_j$ into an equivalent form by using a substitution

$$x_i = \sum_{k=1}^{n} l_{ik}x'_k \qquad (i = 1, 2, \dots, n)$$

In the transformed form $\sum\sum a'_{ij}x_ix_j$, one will have

$$a'_{kk} = M_k \qquad (k = 1, 2, \dots, n)$$

The form $\sum\sum a'_{ij}x_ix_j$ obtained is said to be reduced with respect to the consecutive minima.

The binary, ternary, and quaternary positive quadratic forms can be reduced with respect to the consecutive minima.[‡] The algorithm which one uses in doing the reduction of these forms is founded on the following theorem.

For a positive quadratic form

$$f(x_1, x_2, \dots, x_n) = \sum\sum a_{ij}x_ix_j \quad (n = 2, 3, 4)$$

to be reduced with respect to the consecutive minima, it is necessary and sufficient that one has the inequalities

$$f(x_1, x_2, \dots, x_{k-1}, 1, x_{k+1}, \dots, x_n) \geq a_{kk} \quad (k = 1, 2, \dots, n) \tag{2}$$

and

$$a_{11} \leq a_{22} \leq \dots \leq a_{nn} \tag{3}$$

which is valid for integers of the variables

$$x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n \quad (k = 1, 2, \dots, n)$$

By letting

$$x_i = x'_i + \delta_i x'_k \quad \text{where } \delta_k = 0 \text{ and } i = 1, 2, \dots, n \tag{4}$$

one will determine for the given form $f(x_1, x_2, \dots, x_n)$ integers $\delta_1, \dots, \delta_{k-1}, \delta_{k+1}, \dots, \delta_n$ the condition of which the corresponding value $f(\delta_1, \dots, \delta_{k-1}, 1, \delta_{k+1}, \dots, \delta_n)$ would be smallest. By making successively $k = 1, 2, \dots, n$ and repeating the procedure stated, one will always transform the given form with the aid of the substitution (4) into a form which is no different from the reduced form except by a permutation of the coefficients $(n = 2, 3, 4)$

The procedure stated in the general case can not be carried on indefinitely and one will always arrive at an equivalent quadratic form $\sum a'_{ij}x_ix_j$ which verifies the inequalities (2) and (3), but one does not know from the number of variables $n > 4$ whether the coefficients a'_{kk} $(k = 1, 2, \dots, n)$ in the form obtained exhibit a system of consecutive minima, besides: one also does not know whether the reduction of every positive quadratic form with respect to the consecutive minima is possible.

One rids oneself of the described difficulty by changing the notation of system with n consecutive minima into nothing more than considering the systems (1) which verify the equation

$$\omega = \pm 1 .$$

This is the method known as Hermite method [†] which has recently been improved by Mr. Minkowski in the memoir titled *Diskontinuitätsbereich für arithmetische Aquivalenz. [Discontinuity domain for arithmetical equivalence]* [‡] in the set E, the quadratique $\sum\sum a_{ij}x_ix_j$ being positive and $\alpha_1, \alpha_2, , \alpha_n$ any arbitrary parameters.

In the case $n = 2$, the problem put forward has been solved by Lejeune-Dirichlet and by Hermite [§]

By reflecting upon the principles which have served as basis in these researches of these two illustrious geometers, I have observed that the problem introduced is intimately connected to the problem of the reduction of positive quadratic form.

In effect, Lejeune-Dirichlet and Hermite have demonstrated the following theorem.

The conditions necessary and sufficient for which the inequality

$$ax^2 + 2bxy + cy^2 + 2\alpha y + 2\beta y \geq 0$$

holds, for any integer values of x *and* y*, in general come down to six inequalities*

$$\begin{cases} al^2 + 2blm + cm^2 \pm 2(\alpha l + \beta m) \geq 0, \\ al'^2 + 2bl'm' + cm'^2 \pm 2(\alpha l' + \beta m') \geq 0, \\ al''^2 + 2bl''m'' + cm''^2 \pm 2(\alpha l'' + \beta m'') \geq 0, \end{cases} \tag{5}$$

where the systems of integers

$$(l, m), (l', m') \text{ and } (l'', m'')$$

[‡] *Korkine* and *Zolotareff,* Sur les formes quadratiques positives. (*Mathematische Annalen,* V. 6, p. 336 and V. 11, p. 242)

[†] *Hermite*, Extraits de lettres a Jacobi sur différents objet de la théorie des nombres (This Journal, V. 40, p. 302)

[‡] This Journal, V. 129, p.220

[§] *Lejeune-Dirichlet*, Mémoire cited

Hermite, Sur la théorie des formes quadratique ternaires [On the theory of ternary quadratic forms] (This Journal, V. 40, p. 178) *depend only on coefficients of the quadratic form (a, b, c).*

By considering the parameters α and β as the Cartesian coordinates of a point (α, β) of the plane, one will determine by the inequalities (5) a hexagonal P which is formed by three pairs of parallel edges. The study of properties of the hexagon P plays an important role in the study of Lejeune-Dirichlet which has indicated two fundamental properties of the hexagon P.

I. There exists a group of translations of the hexagon P with the aid of which all the plane will be covered by the congruent hexagons.

II. Any binary positive quadratic form can be transformed by an equivalent form (a, b, c) satisfying the conditions

$$a - b \geq 0, b \geq 0, c - b \geq 0. \tag{6}$$

The hexagon P corresponding to the form (a, b, c), in the case

$$a - b > 0, b > 0, c - b > 0,$$

is characterised by the systems

$$(1, 0), (0, 1), (1, -1) \tag{7}$$

In the case $a - b = 0$, or $(b = 0)$, or $c - b = 0$, the hexagon P reduces itself into a parallelogram.

The inequalities (6) define a domain D of binary quadratic forms which is perfectly determined by the systems (7).

With the help of the substitution

$$x = x', y = -y',$$

one will transform the domain D by a domain D' defined by the inequalities

$$a + b \geq 0, -b \geq 0, c + b \geq 0 \tag{8}$$

which is characterised by the systems

$$(1, 0), (0, 1), (1, 1)$$

One calls reduced by Selling['s method] the binary positive quadratic forms which verify the inequalities (8).‡

By effecting all the transformations of the domain D with the help of substitutions

$$x = px' + qy', y = p'x' + q'y'$$

of integer coefficients and of determinant ±1, one obtains a set (D) of domains of binary quadratic forms.

The set (D) of domains uniformly partitions the set of all the binary positive quadratic forms, that is to say: a form which is interior to any one domain D of the set (D) does not belong to any other domain of this set; a form which is interior to a face of the domain D belongs to only one other domain of the set (D) which is contiguous to the domain (D) by this face.

The results summarised have brought me to a new point of view on the problem of reduction of positive quadratic forms.

The problem of reduction of positive quadratic forms consist of a uniform partition of the set of positive quadratic forms with the help of domains of forms, determined using linear inequalities and enjoying the property that any substitution of integer coefficients and of determinant ±1 does not change the set (D) of these domains. By partitioning the set (D) into classes of equivalent domains and by choosing the representatives of all the classes

$$D, D_1, \ldots, D_{m-1}, \tag{9}$$

one will call reduced the quadratic forms which belong to these domains.

One could attach the supplementary condition to the domains (9) by demanding: 1). that $m = 1, 2$, 2). that the positive quadratic forms interior to the domain D are not equivalent and lastly, 3). that the number of linear inequalities which define the domain D be the smallest one possible.

I hope to return another time to the problem posed of the reduction of positive quadratic forms.

In this mémoire, I restrict myself to the study of domains of quadratic forms which one obtains by generalising the results shown in studies of Lejeune-Dirichlet and of Hermite for the positive quadratic forms in any number of variables.

The hexagon of Lejeune-Dirichlet can be replaced for the positive quadratic forms of n variables by a convex polyhedron of the analytical space in n dimensions.

For a positive quadratic form $\sum\sum a_{ij}x_ix_j$, the corresponding polyhedron R presents a set of points (α_i) verifying the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq 0 \tag{10}$$

in the set E. The polyhedron R can be determined with the help of independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} \geq 0, \; (k = 1, 2, \ldots, \tau)$$

the number 2τ of which does not exceed a limit

$$2\tau \leq 2(2^n - 1).$$

the systems of integers

$$\pm(l_{11}, l_{21}, \ldots, l_{n1}), \pm(l_{12}, l_{22}, \ldots, l_{n2}), \ldots, \pm(l_{1\tau}, l_{2\tau}, \ldots, l_{n\tau}) \tag{11}$$

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‡ *Selling,* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p.143)

define by the corresponding equations

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} = 0$$

2τ faces in $n-1$ dimensions of the polyhedron R. As these faces partition themselves into τ pairs of parallel faces, I call parallelohedron the polyhedron R corresponding to any positive quadratic form.

The systems (11) enjoy many important properties.

1. For a system $(l_1, l_2, \dots, l_n)$ to belong to the series (11), it is necessary and sufficient that two systems $(l_1, l_2, \dots, l_n)$ and $(-l_1, -l_2, \dots, -l_n)$ are the only representations of the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set composed of all the systems of integers which are congruent to the system $(l_1, l_2, \dots, l_n)$ by relation to the modulus 2, the system $l_1 = 0, l_2 = 0, \dots, l_n = 0$ being excluded.

2. Among the systems (11) are found all the representations of the arithmetical minimum of the positive quadratic form $\sum\sum a_{ij}x_ix_j$.

3. among the systems (11) are found all the systems (1) which represent n consecutive minima of the form $a_{ij}x_ix_j$.

4. All the determinants which one can form of any n systems belonging to the series (11) do not exceed in numerical value a limit $n!$.

By designating by the symbol S_ν the number of faces in ν dimensions ($\nu = 0, 1, 2, \dots, n-1$) of a parallelohedron R, I have found that

$$S_\nu \leq (n+1-\nu)\Delta^{(n=\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \dots, n-1)$$

By making $\nu = 0$ in this inequality, one obtains

$$S_0 \leq (n+1)!\,,$$

therefore the number of vertices of a parallelohedron R does not exceed a limit $(n+1)!$. By making $\nu = n-1$, one obtains

$$S_{n-1} \leq 2(2^n - 1)\,.$$

I demonstrate in this memoir that there exist parallelohedra, the symbol S_ν for which are expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \dots, n-1)$$

All these parallelohedra are primitive.

The notation of positive parallelohedra plays an important role in my studies.

I have arrived at the notation of primitive parallelohedra by observing that the parallelohedra possess Property I of hexagons of Lejeune-Dirichlet, in knowing:

I. There exists a group of transformations of a parallelohedron R with the help of which one uniformly fills the analytical space in n dimensions by the congruent parallelohedra.

Designate by (R) the set of parallelohedra which are defined by the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i\,,$$

$l_1, l_2, \dots, l_n$ being arbitrary integers. Ay system (l_I) of integers characterise a parallelohedron of the set (R).

I demonstrate that the set (R) of parallelohedra corresponding to the various systems (l_i) of integers unformly fills the space in n dimensions.

The corresponding group of translations of the parallelohedron R defined by the inequalities (10) is composed of vectors $[\lambda_i]$ which are determined by the equalities

$$\lambda_i = -\sum_{k=1}^{n} a_{ik}l_k, \quad (i = 1, 2, \dots, n)$$

$l_1, l_2, \dots, l_n$ being arbitrary integers.

Any vertex (α_i) of parallelohedra of the set (R) belongs to at least $n+1$ parallelohedra. I call simple a vertex (α_i) which belongs only to $n+1$ parallelohedra of the set (R) and I establish a notion of primitive parallelohedron as follows:

One call primitive parallelohedron, a parallelohedron the vertices of which are simple.

All the parallelohedra which are not primitive are called nonprimitive. From this point of view, the hexagon of Lejeune-Dirichlet presents a primitive parallelohedron and each parallelogram is a nonprimitive parallelohedron in two dimensions.

Any nonprimitive parallelohedron is a boundary of primitive parallelohedra and can be considered as a case of degeneracy of primitive parallelohedra.

I divide the primitive parallelohedra into various types by characterising a type of primitive parallelohedra by a set (L) of simplexes correlative to the various vertices of parallelohedra which belong to the set (R).

An identical vertex (α_i) is determined by $n+1$ equations

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = A. \quad (k = 0, 1, 2, \dots, n)$$

In $n+1$ systems of integers

$$(l_{1k}, l_{2k}, \dots, l_{nk}), \quad (k = 0, 1, 2, \dots, n)$$

I make a simplex L correspond by defining it as a set of points which are determined by the equations

$$x_i = \sum_{k=0}^{n} \vartheta_k l_{ik}, \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0.$$

$$(k=0,1,2,\dots,n, i=1,2,\dots,n)$$

The set (L) of simplexes which are correlative to the vertices of the set (R) of primitive parallelohedra enjoys important properties.

1. The set (L) of simplexes uniformly partition the space of n dimensions.

2. By effecting the various translations of a simplex of the set (L) the length of vector $[l_i]$ which are determined by the arbitrary integers $l_1, l_2, \dots, l_n$, one obtains a class of congruent simplexes which belong to the set (L).

3. The number of incongruent simplexes of the set (L) is finite.

Property II of hexagons of Lejeune-Dirichlet for the primitive parallelohedra can be generalised as follows:

II. All the quadratic forms which define the primitive parallelohedra belonging to the type characterised by the set (L) of simplexes are interior to a domain of quadratic form in $\frac{n(n+1)}{2}$ dimensions defined by linear inequalities.

I obtains the linear inequalities which define a domain D of quadratic forms corresponding to a set (L) of simplexes by examining the incongruent edges of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes.

An vertex (α_i) of primitive parallelohedra of the set (R) belongs to $n+1$ edges $[\alpha_i, \alpha_{ik}]$ of these parallelohedra $(k=0,1,2,\dots,n)$.

By putting

$$\alpha_{ik} - \alpha_i = p_{ik}\rho_k, \quad (i=1,2,\dots,n; k=0,1,2,\dots,n)$$

one can determine the positive parameter ρ_k, of such manner that the numbers $p_{1k}, p_{2k}, \dots, p_{nk}$ are integers and do not possesses common divisor. I demonstrate that the parameter ρ_k expressed by a linear function

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij} \tag{12}$$

of coefficients of the given quadratic form $\sum\sum a_{ij}x_ix_j$, the coefficients

$$p_{ij}^{(k)} = p_{jk}^{(k)}, \quad (i=1,2,\dots,n; j=1,2,\dots,n)$$

being rational.

I call regulator of the edge $[\alpha_i, \alpha_{ik}]$, the function ρ_k determined by the formula (12); the system (p_{ik}) is called characteristic of the edge

$$[\alpha_i, \alpha_{ik}]. \quad (k=0,1,2,\dots,n)$$

As the edge $[\alpha_i, \alpha_{ik}]$ is correlative to a face P_k of $n-1$ dimensions of the simplex L which is correlative to the vertex (α_i), I call the function (12) *regulator of the face P_k* and the system $\pm(P_{ik})$ *characteristic of the face P_k of the simplex L* $(k=0,1,2,\dots,n)$

By designating by

$$\rho_k \text{ and } \pm(p_{ik}), \quad (k=1,2,\dots,\sigma)$$

the regulators and the characteristics of all the incongruent faces in $n-1$ dimensions of the set (L) of simplexes, I demonstrate the following important theorem:

The domain of quadratic forms which is characterised by the set (L) of simplexes is defined by the linear inequalities

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij} \geq 0. \quad (k=1,2,\dots,\sigma)$$

All the domains of quadratic forms which I have studied in this memoir possess a remarkable property: they are simple domains, that is to say the number of independent inequalities which define them is equal to $\frac{n(n+1)}{2}$.

Another coincidence has attracted my attention for a long time: that is the relation which exists within the results shown in this memoir and those which have been obtained in my first memoir titled: "On some properties of perfected positive quadratic forms" † I have observed that the set of characteristics $\pm(p_{ik}), k=1,2,\dots,\sigma$ is nothing but the set of all the representations of the minimum of a perfect quadratic form φ. Thee domain D either coincides well with the domain R corresponding to the perfect form φ, or presents well a group of this domain.

Despite all my effort, I have not succeeded in discovering the tie which attaches the two problems shown and which seem to be so different, abstraction made of a remarkable formula

$$\sum\sum a_{ij}x_ix_j = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma} \rho_k\omega_k(p_{1k}x_1 + p_{2k}x_2 + \cdots + p_{nk}x_n)^2$$

which supplies the expression of an arbitrary quadratic form $\sum\sum a_{ij}\cdot x_ix_j$ in function of the regulators ρ_k $(k=1,2,\dots,\sigma)$ which are determined by the formula (12).

In this formula ω_k $(k=1,2,\dots,\sigma)$ are positive integers which depend only on corresponding faces of simplexes of the set (L).

To the various types of primitive parallelohedra corresponds a set (D) of domains of quadratic forms. The set (D) uniformly partitions the set of all the positive quadratic forms in n variables.

I show in this mémoire an algorithm, by the aid of which one can determine all the domains of forms which are contiguous to a domain of the set (D) by the faces in $\frac{n(n+1)}{2}-1$ dimensions. This algorithm comes down to a certain reconstruction of the set (L) of simplexes by another set (L').

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† This Journal, V. 133, p. 97

The set (D) of domains of forms transforms into itself by all the substitutions of integer coefficients and of determinant ± 1. By dividing the set (D) into classes of equivalent domains, one obtains with the aid of the algorithm shown the representatives

$$D, D_1, \ldots, D_{m-1}$$

of various classes of domains belonging to the set (D).

By calling reduced the quadratic forms which belong to the domains obtained, one establishes a new method of reduction of positive quadratic forms.

I have applied the general theory shown to the study of two types of primitive parallelohedra of the space in n dimensions which correspond to the principal domain of quadratic forms and to the domains which are contiguous to the principal domain by the faces in $\frac{n(n+1)}{2} - 1$ dimensions. The principal domain is defined by the inequalities

$$\sum_{k=1}^{n} a_{ik} \geq 0, \quad (i = 1, 2, \ldots, n)$$

$$-a_{ij} \geq 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

I study in detail the parallelohedra of the space in 2, 3 and 4 dimensions.

In the space in 2 dimensions, there is only one type of primitive parallelohedra, provided that one does not consider as different the equivalent types; it is the hexagon of Lejeune-Dirichlet.

The set (D) of domains is composed in this case of a single class, the representative of which is the principal domain defined by the inequalities (8).

In the space in 2 dimensions, there is only one single space of primitive parallelohedra – it is the parallelogram.

In the space in 3 dimensions, there is only one single type of primitive parallelohedra – it is a polyhedron of 14 faces, 8 of which are hexagonal and 6 of which are parallelogrammatic.

The set (D) of domains is composed in this case of a single class, the representative of which is the principal domain. By calling reduced a ternary positive quadratic form $ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy$ which belongs to the principal domain determined with the help of inequalities

$$a + b' + b'' \geq 0, a' + b'' + b \geq 0, a'' + b + b' \geq 0, -b \geq 0, -b' \geq 0, -b'' \geq 0,$$

one will arrive at the method of reduction of ternary positive quadratic forms due to Selling. ‡

In the space in 3 dimensions, there are 4 spaces of primitive parallelohedra, they are : 1). the parallelepiped, 2). the prism of hexagonal base, 3). the parallelegrammatic dodecahedron and 4). the dodecahedron in 4 hexagonal faces and 8 parallelogrammatic faces.

In the space of 4 dimensions, there are three types of primitive parallelohedra. The set (D) of domains is composed of three classes of domains of quaternary quadratic forms.

I have determined the three representatives of these classes

$$D, D', D'',$$

By calling as reduced the quaternary positive quadratic form which belong to the domains D, D', D'', I have arrived at a modification of the methods of reduction of quaternary positive quadratic forms due to Mr. Charve. †

By virtue of this theorem, the problem of uniform partition of the space in n dimensions by congruent primitive parallelohedra always comes down to the study of parallelohedra corresponding to the positive quadratic forms.

I am inclined to think, without being able to demonstrate, that the theorem introduced is also true for the nonprimitive parallelohedra.

The parallelohedra of the space in 2 and in 3 dimensions have been studied by Mr. Fedorow ¶ which has discovered with the help of purely geometrical considerations, the exitence of two spaces of parallelohedra in the space in 2 dimensions and the existence of five spaces of parallelohedra in the space in 3 dimensions. Mr. Fedorow has demonstrated that there is no other parallelohedra in the space of 2 and of 3 dimensions.

The parallelohedra in 3 dimensions of Mr. Fedorow play an important role in the theory of the structure of crystals. §

First part

Uniform partition of the analytical space in n dimensions with the aid of translations of the same convex polyhedron

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‡ *Selling,* Mémoire cited

† *Charve,* De la réduction des formes quadratiques quaternaires positives [Of the reduction of quaternary quadratic forms] (Comptes-Rendus des séances de l'Academie du Paris), V. 92, p.782 and Annales de l'École Normale supérieure, 2^{nd} serie, V. XI, p.119

¶ *Fedorow,* Basic principles in the theory of diagrams. St. Petersbourg, 1885 (in Russian)

Fedorow, Reguläre Plan- und Raumteilung. [Regular planar and space partition] (Abhandlungen der K. bayer. Akademie der Wiss. II Cl., XX Bd. II Abt. München, 1899)

See also: *Minkowski,* Allgemeine Lehrsätze über die convexen Polyeder. [General theorems on the onvex polyhedron] (Nachrichten von der Köigl. Gesellschaft der Wissenschaften zu Göttingen, Matheem. -Physikalische Klasses, 1897, p.198)

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Section I
General properties of parallelohedra

On the convex polyhedra in n dimensions

1 One will call point of the analytical space in n dimensions any systems $(x_1, x_2, \ldots, x_n)$, or simply (x_i), of real values of variables $x_1, x_2, \ldots, x_n$.

Consider a system of linear inequalities

$$a_{0k} + \sum_{i=1}^{n} a_{ik} x_i \geq 0 \quad (k = 1, 2, \ldots, \sigma) \tag{1}$$

of any real coefficients.

One will say that the set R of points verifying the inequalities (1) is of n dimensions, if there exist points satisfying the conditions

$$a_{0k} + \sum a_{ik} x_i > 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call them point, interior to the set R.

Fundamental principle. † *For the set R of points verifying the inequalities (1) to be of n dimensions, it is necessary and sufficient that the equation*

$$\rho_0 + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} x_i) = 0$$

does not reduce into an identity so long as all the parameters $\rho_0, \rho_1, \ldots, \rho_\sigma$ are positive or zero.

Definition I. One will call convex polyhedron any set of points verifying a system of linear inequalities, on condition that this set be bounded and of n dimensions.

2 Let us suppose that the inequalities (1) define a convex polyhedron R and suppose that all the inequalities (1) be independent. In such case, the polyhedron R possesses σ faces in $n-1$ dimensions which are defined by the corresponding equations

$$a_{0k} + \sum a_{ik} x_i = 0. \quad (k = 1, 2, \ldots, \sigma)$$

Definition II. Suppose that a point (α_i) belonging to R verifies the equations

$$a_{0r} + \sum a_{ir} x_i = 0, \quad (r = 1, 2, \ldots, \mu) \tag{2}$$

and that one had the inequalities

$$a_{0k} + \sum a_{ik} x_i > 0. \quad (k = \mu + 1, \ldots, \sigma)$$

Designate by ν the number of dimensions of the set $P(\nu)$ composed of points belonging to R and verifying the equations (2). One will call face in ν dimensions of the polyhedron R *the set* $P(\nu)a$, $(\nu = 0, 1, 2, \ldots, n-1)$.

In the case $\nu = 1$, one will call edge of the polyhedron R a face $P(1)$ and in the case $\nu = 0$, one will call vertex of the polyhedron a face $P(0)$.

For more generality in the notations, one will designate by the symbol $P(n)$ the polyhedron R itself.

Under this restriction, one can introduce the following proposition:

Any point belonging to the polyhedron R is interior to a face $P(\nu)$ of that polyhedron, where $\nu = 0, 1, 2, \ldots, n$.

3 Let us suppose that the polyhedron R possesses s vertices

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is}).$$

Designate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{im})$$

all the vertices of R which verify the equations (2).

Theorem. † *The face $P(\nu)$ in ν dimensions $(\nu = 0, 1, 2, \ldots, n)$ of the polyhedron R defined by the equations (2) presents a set of points determined by the aid of equalities*

$$x_i = \sum_{r=1}^{m} \vartheta_r \alpha_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r \geq 0. \quad (r = 1, 2, \ldots, m)$$

Set of domains in n dimensions corresponding to the different vertices of a convex polyhedron.

4 Let us suppose a vertex (α_i) of the polyhedron R be determined by the equations

$$a_{0k} + \sum a_{ik} x_i = 0. \quad (k = 1, 2, \ldots, \mu) \tag{1}$$

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† The principle announced differs only in the formulation from the fundamental principle explained in my first memoir titled: On some properties of perfect positive quadratic forms. (This journal, V. 133, p. 113)

† See my mémoire cited, Number 12

Definition. One will call domain corresponding to the vertex (x_i) the set A of points determined with the help of equalities

$$x_i = \sum_{k=1}^{\mu} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, \mu) \tag{2}$$

Designate by

$$A_1, A_2, \ldots, A_s \tag{3}$$

the domains corresponding to the different vertices

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

of the polyhedron R. By virtue of the definition established, the set (3) of domains enjoys the following properties:

I. All the domains of the set (3) are in n dimensions.

Let us suppose the domain A determined by the equalities (2) be not in n dimensions.

All the points (x_i) belonging to the domain A verify at least one linear equation

$$\sum p_i x_i = 0.$$

By virtue of (2), one will have

$$\sum p_i a_{ik} = 0. \ (k = 1, 2, \ldots, \mu) \tag{4}$$

As the equations (1) define a vertex (α_i) of the polyhedron R, one will find among the systems

$$(a_{11}, \ldots, a_{n1}), (a_{12}, \ldots, a_{n2}), \ldots, (a_{1\mu}, \ldots, a_{n\mu})$$

n systems the determinant of which is not zero; it follow that the equalities (4) are impossible.

II. Any point of the space in n dimensions belong to at least one domain of the set (3).

Let (a_i) be an arbitrary point. Examine the sum

$$\sum a_i \alpha_{ik}, \quad (k = 1, 2, \ldots, s)$$

and suppose that the smallest sum $\sum a_i x_i$ correspond to the vertex (α_i) defined by the equations (1). One will have the inequalities

$$\sum a_i \alpha_{ik} \geq \sum a_i \alpha_i. \quad (k = 1, 2, \ldots, s)$$

By virtue of the theorem of Number 3, one obtains

$$\sum a_i x_i \geq \sum a_i \alpha_i,$$

for any point (x_i) belonging to the polyhedron R.

One concludes that the inequalities

$$\sum a_i \alpha_i - \sum a_i x \geq 0 \text{ and } a_{0k} + \sum a_{ik} x_i \geq 0 \ (k = 1, 2, \ldots, \sigma)$$

can not define a polyhedra in n dimensions and, by virtue of the fundamental principle of Number 1, one will have an identity

$$\rho_0 + \rho(\sum a_i \alpha_i - \sum a_i x_i) + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} x_i) = 0,$$

where

$$\rho_0 \geq 0, \rho \geq 0, \rho_k \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By making $x_i = \alpha_i$ in this identity, it will become

$$\rho_0 + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} \alpha_i) = 0,$$

and as according to the supposition made

$$a_{0k} + \sum a_{ik} \alpha_i > 0$$

as long as $k = \mu + 1, \ldots, \sigma$, it is necessary that

$$\rho_0 = 0, \rho_{\mu+1} = 0, \ldots, \rho_\sigma = 0,$$

therefore

$$\rho(\sum a_i \alpha_i - \sum a_i x_i) + \sum_{k=1}^{\mu} \rho_k (a_{0k} + \sum a_{ik} x_i) = 0.$$

One draws

$$a_i = \sum_{k=1}^{\mu} \frac{\rho_k}{\rho} a_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0, \ (k = 1, 2, \ldots, \mu)$$

therefore the points (α_i) belongs to the domain A.

III. A point which is interior to a face $A(\nu)\,(\nu = 0, 1, 2, \ldots, n)$ *of any domain of the set (3) belongs only to the domains of the set (3) which are contiguous by the face* $A(\nu)$.

Suppose the point (a_i) be interior to a face $A(\nu)$ of the domain A.

By designating with

$$(a_{i1}), (a_{i2}), \ldots, (a_{i\tau}), \tau \leq \mu$$

the points which characterise the face $A(\nu)$, one can put †

$$a_i = \sum_{k=1}^{\tau} \rho_k a_{ik} \text{ where } \rho_k > 0. \quad (k = 1, 2, \ldots, \tau)$$

Suppose that the point (a_i) be interior to another face $A'(\nu')$ of a domain A' which corresponds to a vertex (α'_i). One can put

$$a_i = \sum_{h=1}^{\tau'} \rho'_h a'_{ih} \text{ where } \rho'_h > 0. \ (h = 1, 2, \ldots, \tau')$$

By virtue of these equalities, one will have an identity

$$\rho_0 + \sum_{h=1}^{\tau} \rho_k (a_{0k} + \sum a_{ik} x_i) = \sum_{h=1}^{\tau'} (a'_{0h} + \sum a'_{ih} x_i) \tag{5}$$

By making within this identity $x_i = \alpha_i$, one obtains

$$\rho_0 = \sum_{h=1}^{\tau'} (a'_{0h} + \sum a'_{ih} \alpha_i),$$

and it results that

$$\rho_0 \geq 0.$$

By making within the identity (5) $x_i = x'_i$, one obtains

$$\rho_0 + \sum_{k=1}^{\tau} \rho_k (a_{0k} + \sum a_{ik} \alpha'_i) = 0;$$

consequently $\rho_0 = 0$ and

$$a_{0k} + \sum a_{ik} \alpha'_i = 0. \quad (k = 1, 2, \ldots, \tau)$$

In the same manner, one finds

$$a'_{0h} + \sum a'_{ih} \alpha_i = 0. \quad (h = 1, 2, \ldots, \tau')$$

One concludes that the two faces $A(\nu)$ and $A'(\nu')$ coincide. ‡

By virtue of properties demonstrated of the set (3) of domains, one will say that this set uniformly *partitions* the space in n dimensions.

Definition of the group of vectors

5 *Definition I. One will call vector the set of points determined with the help of equalities*

$$x_i = x_i + u(\alpha'_i - \alpha_i) \text{ where } 9 \leq u \leq 1, \tag{1}$$

(α_i) *and* (α'_i) *being any two different points.*

One will designate the vector determined with the help of equalities (1) by the symbol $[\alpha_i, \alpha'_i]$. In this case $\alpha_i = 0\,(i = 1, 2, \ldots, n)$, one will designate the corresponding vector by the symbol $[\alpha'_i]$ and one will call it vector of the point (α'_i).

Definition II. Suppose that

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}] \tag{2}$$

be the vectors of arbitrary points $(\lambda_{i1}), (\lambda_{i2}), \ldots, (\lambda_{im})$. *One will call* group of vectors *the set* G *of vectors determined with the help of equalities*

$$\lambda_i = \sum_{k=1}^{m} l_k \lambda_{ik},$$

$l_1, l_2, \ldots, l_m$ *being of arbitrary integers.*

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† See my mémoire cited, Number 13

‡ See my mémoire cited, Number 20, p. 133

One will call *basis of the group G of vectors* the vectors (2).

Translation of polyhedra.

6 *Definition. Effect a linear transformation of a polyhedron R with the help of a substitution*

$$x_i = x'_i - \lambda_i, \ (i = 1, 2, \dots, n) \tag{1}$$

the coefficients $\lambda_1, \lambda_2, \dots, \lambda_n$ being arbitrary. One will say that one has effected a translation of the polyhedron R the length of the vector $[\lambda_i]$.

Suppose that the polyhedron R be determined by the inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0, \quad (k = 1, 2, \dots, \sigma)$$

The transformed polyhedron R' will be determined, by virtue of (1), by the inequalities

$$a_{0k} + \sum a_{ik}(x_i - \lambda_i) \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

One will call congruent the polyhedra R and R'.

7 Let G be a group of vectors. By effecting the different translations of the poly hedron R the length of vectors belonging to the group G, one will form a set R of congruent polyhedra.

One will say that the set (R) of congruent polyhedra uniformly partition the space in n dimensions in the following conditions.

I. Any point of the space in n dimensions belongs to at least one polyhedron of the set (R).

II. A point which is interior to any one face $P(\nu)$ $(\nu = 0, 1, 2, \dots, n)$ of a polyhedron of the set (R) belongs to only the polyhedrons of the set (R) which are contiguous by the face $P(\nu)$.

Definition of parallelohedra

8 *Definition. One will call parallelohedron any convex polyhedron R possessing a group G of translations with the aid of which one can uniformly fill the space in n dimensions by the polyhedra congruent to the polyhedron R.*

By virtue of the definition established, the parallelohedra possess an important property, in knowing:

By effecting a linear transformation of a parallelohedron with the help of a substitution by any real coefficients

$$x_i = \alpha_{i0} + \sum_{k=1}^{n} \alpha_{ik} x'_k, \quad (i = 1, 2, \dots, n)$$

one obtains a convex polyhedron which is also a parallelohedron.

Observe that by virtue of the definition established, any parallelohedron of the space in n dimensions is a parallelohedron.

Properties of the group of vectors of a parallelohedron.

9 Suppose that a parallelohedron R be defined by the inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

Designate by G the group of the parallelohedron R and suppose that the group G possesses the basis

$$[\lambda_{i1}], [\lambda_{i2}], \dots, [\lambda_{im}]. \tag{2}$$

All the vectors which form the basis of the group G can not verify the same linear equation

$$\sum p_i \lambda_i = 0,$$

because otherwise the set (R) of congruent parallelohedra corresponding to the group G would not fill the space in n dimensions.

One concludes that among the vectors (2) there are n vectors

$$[\lambda_{i1}], [\lambda_{i2}], \dots, [\lambda_{im}] \tag{3}$$

the determinant $\pm\Delta$ of which is not zero; one will call them independent.

Theorem I. The numerical value Δ of the determinant of n independent vectors possesses a limit

$$\Delta \geq \int_{(R)} dx_1 dx_2 \cdots dx_n \,.$$

Let (α_i) be any point which is interior to the parallelohedron R. Introduce within our researches a parallelepiped K determined with the aid of equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \quad (i = 1, 2, \dots, n) \tag{4}$$

where

$$-\delta \leq u_k \leq \delta. \quad (k = 1, 2, \dots, n) \tag{5}$$

One can choose the positive parameter δ in such manner that all the points of the parallelohedron R defined by the inequalities (1) belong to the parallelepiped K.

Take a positive integer m and determine $(m+1)^n$ systems $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequalities

$$0 \leq l_k \leq m. \quad (k = 1, 2, \ldots, n) \tag{6}$$

Designate by

$$\lambda_i^{(h)} = \sum_{k=1}^{n} l_k^{(h)} \lambda_{ik}, \quad (h = 1, 2, \ldots, (m+1)^n) \tag{7}$$

$(m+1)^n$ corresponding vectors belonging to the group G.

By applying the translations of the parallelohedra R the length of vectors (7), one obtains $(m+1)^n$ different parallelohedra of the set (R):

$$R^{(h)}. \quad (h = 1, 2, \ldots, (m+1)^n) \tag{8}$$

Designate by H a parallelepiped which is determined by the equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \quad (i = 1, 2, \ldots, n) \tag{9}$$

where

$$-\delta \leq u_k \leq m + \delta. \quad (k = 1, 2, \ldots, n) \tag{10}$$

I argue that all the points of parallelohedron (8) belong to the parallelepiped H. In effect, let $(x_i^{(h)})$ be any point of the parallelohedron $R^{(h)}$ $(h = 1, 2, \ldots, (m+1)^n)$. By posing

$$x_i = x_i^{(h)} - \lambda_i^{(h)}, \quad (i = 1, 2, \ldots, n) \tag{11}$$

one obtains a point (x_i) belonging to the parallelohedron R which is congruent to the point given $(x_i^{(h)})$. By virtue of (4), (7) and (11), one obtains

$$x_i^{(h)} = \alpha_i + \sum_{k=1}^{n} (l_k + u_k) \lambda_{ik},$$

and by (5) and (6), it becomes

$$-\delta \leq l_k + u_k \leq m + \delta, \quad (k = 1, 2, \ldots, n)$$

thus the point $(x_i^{(h)})$ belongs to the parallelepiped H.

It follows that

$$\int_{(H)} dx_1 dx_2 \cdots dx_n \geq \sum_h \int_{(R^h)} dx_1 dx_2 \cdots dx_n. \ (h = 1, 2, \ldots, (m+1)^n)$$

By observing that

$$\int_{(H)} dx_1 dx_2 \cdots dx_n = \Delta (m + 2\delta)^n$$

and that

$$\int_{(R^h)} dx_1 dx_2 \cdots dx_n = \int_{(R)} dx_1 dx_2 \cdots dx_n, \ (h = 1, 2, \ldots, (m+1)^n)$$

one obtains

$$\Delta (m + 2\delta)^n \geq (m+1)^n \int_{(R)} dx_1 dx_2 \cdots dx_n.$$

By making the number m increase indefinitely, one finds

$$\Delta \geq \int_{(R)} dx_1 dx_2 \cdots dx_n.$$

10

Theorem II. The group G of vectors of a parallelohedron possesses basis formed by n independent vectors.

Designate by G' a group of vectors having the basis (3). It can be that the two groups G and G' coincide. In that case n vectors (3) present a basis of the group G.

By supposing the contrary, one will have among the vectors (2) at least one vector $[\lambda_i']$ which does not belong to the group G'. By putting

$$\lambda_i' = \sum_{k=1}^{n} l_k' \lambda_{ik},$$

one will have among the numbers $l_1', l_2', \ldots, l_n'$ at least one number which is fractional.

Designate by $l_1, l_2, \ldots, l_n$ the integers verifying the inequalities

$$\left| l_k' - l_k \right| \leq \frac{1}{2} \quad (k = 1, 2, \ldots, n)$$

and suppose that $l'_r - l_r \neq 0$.

By designating

$$\lambda'_{ik} = \lambda_{ik}\,,\ (k = 1, 2, \ldots, n, k \neq r) \text{ and } \lambda'_{ir} = \lambda'_i - \sum l_k \lambda_{ik},\,,$$

one obtains a system of n independent vectors

$$[\lambda'_{i1}], [\lambda'_{i2}], \ldots, [\lambda'_{in}]$$

belonging to the group G, the determinant $\pm\Delta'$ of which verifies the inequality

$$0 < \Delta' \leq \frac{1}{2}\Delta.$$

The procedure explained can not be prolonged indefinitely, by virtue of Theorem I, therefore one will always obtain a system of n vectors forming the basis of the group G.

11 *Theorem III. The numerical value Δ of the determinant of a system of n vectors forming the basis of the group G is expressed by the formula*

$$\Delta = \int_{(R)} dx_1 \cdot dx_2 \cdots dx_n\,.$$

suppose that the system (3) of n vectors presents a basis of the group G.

Introduce in okur studies a parallelepiped H' determined with the help of equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik},\ (i = 1, 2, \ldots, n) \tag{12}$$

where

$$\delta \leq u_k \leq m - \delta\,.\,(k = 1, 2, \ldots, n) \tag{13}$$

I argue that any point of the parallelepiped H' belongs to at least one of parallelohedron (8). In effect, let (x'_i) be any point of the parallelepiped H'.

Designate by R^0 a parallelohedron of the set (R) to which belongs the point (x'_i). Let $[\lambda_i]$ be the vector which defines a translation of the parallelohedron R to R^0. By putting

$$x_i = x'_i - \lambda_i\,, \tag{14}$$

one obtains a point (x_i) belonging to the parallelohedron R which is congruent to the point x'_i. By virtue of the supposition made, the vector $[\lambda_i]$ can be determined by the equalities

$$\lambda_i = \sum_{k=1}^{n} l_k \lambda_{ik}\,. \tag{15}$$

As the point (x'_i) belongs to the parallelepiped H', one will present the equalities (14), by (12) and (15), in the following form:

$$x + i = \alpha_i + \sum_{k=1}^{n} (u_k - l_k)\lambda_{ik}\,.$$

The point (x_i) belonging to the parallelohedron R belongs also, by virtue of the supposition made, to the parallelepiped K determined by the equalities (4), by condition of (5). It follows that

$$-\delta \leq u_k - l_k \leq \delta\,, \quad (k = 1, 2, \ldots, n)$$

and as, by (13),

$$\delta \leq u_k \leq m - \delta\,, \quad (k = 1, 2, \ldots, n)$$

it becomes

$$0 \leq l_k \leq m, \quad (k = 1, 2, \ldots, n)$$

therefore the vector $[\lambda_i]$ determined by the equalities (5) is among the vectors (7) and the point examined (x') of the parallelepiped H' belongs to a parallelohedron of the series (8).

It follow that

$$\int_{H'} dx_1 dx_2 \cdots dx_n \leq \sum_h \int_{(R^h)} dx_1 dx_2 \cdots dx_n\,.$$

By making the number m grow indefinitely, one obtains

$$\Delta \leq \int_{(R)} dx_1 dx_2 \cdots dx_n\,.$$

By virtue of Theorem I, it is necessary that

$$\Delta = \int_{(R)} dx_1 dx_2 \cdots dx_n\,.$$

Properties of faces in $n-1$ dimensions of a parallelohedron.

12

Suppose that a parallelohedron R be defined by the independent inequalities

$$a_{0k} + \sum a_{ik}x_i \geq 0\,. \quad (k = 1, 2, \ldots, \delta)$$

Designate by $P_k (k = 1, 2, \ldots, \sigma)$ the faces in $n-1$ dimensions of the parallelohedron R determined by the corresponding equations

$$a_{0k} + \sum a_{ik}x_i = 0\,. \tag{1}$$

Let (α_i) be a point which is interior to the face P_k. Examine a parallelepiped K defined by the equalities

$$x_i = \alpha_i + u_i \text{ where } |u_i| \leq \epsilon\,. (i = 1, 2, \ldots, n) \tag{2}$$

One can choose a parameter ϵ however small that one will have the inequalities

$$a_{0r} + \sum a_{ir}x_i > 0,\ (r = 1, 2, \ldots, \sigma, r \neq k) \tag{3}$$

for any point (x_i) of the parallelepiped K. It results in that all the points of the parallelepiped K verifying the inequality

$$a_{0k} + \sum a_{ik}x_i \geq 0 \tag{4}$$

belong to the parallelohedron R. As the point (α_i) verifies the equation (1), the equation (4) reduces, by reason of (2), to this one here

$$\sum a_{0k}u_i \geq 0\,.$$

I argue that one can choose a value of the parameter ϵ however sall that all the points of the parallelepiped K verifying the inequality

$$\sum a_{ik}u_i \leq 0$$

will belong to another parallelohedron R_k of the set (R). By relying on the demonstrated properties of the group G of vectors, one will easily demonstrate the proposition stated.

Two parallelohedra R and R_k are contiguous by the face P_k in $n-1$ dimensions. Designate by $[\lambda_{ik}]$ the vector which defines a translation of the parallelohedron R_k to R. The face P_k which is defined in the parallelohedron R by the equation (1) will be defined in the parallelohedron R_k by the equation

$$-a_{0k} - \sum a_{ik}x_i = 0\,. \tag{5}$$

By carrying out a translation of the face P_k the length of the vector λ_{ik}, one obtains another face P_k of the parallelohedron R which will be within the parallelohedron determined by the equation

$$-a_{0k} - \sum a_{ik}(x_i - \lambda_{ik}) = 0\,.$$

One will call parallel the faces P_k and P'_k of the parallelohedron R.We have arrived at the following important result:

All the faces in $n-1$ dimensions of a parallelohedron can be divided into pairs of parallel faces.

13

Designate by

$$R_1, R_2, \ldots, R_\sigma$$

all the parallellohedra which are contiguous to the parallelohedron R by the faces $P_1, P_2, \ldots, P_\sigma$. Designate by

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{i\sigma}] \tag{6}$$

the corresponding vectors.

By virtue of the definition of the parallelohedron, the vectors (6) form the basis of the group G. Among the vectors of this group there exist the systems of n vectors which form a basis of the group G.

Congruent faces in different dimensions of a parallelohedron.

14

Suppose that a face $P(\nu)$ in ν dimensions of a parallelohedron R also belongs to the parallelohedra $R_1, R_2, \ldots, R_\tau$ of the set (R). Let (α_i) be a point which is interior to the face $P(\nu)$. One can determine a positive value of the parameter ϵ in such a manner that all the point of the parallelepiped K defined by the equalities

$$x_i = \alpha_i + u_i \text{ where } |u_i| \leq \epsilon \quad (i = 1, 2, \ldots, n)$$

belong to the parallelohedra $R, R_1, R_2, \ldots, R_\tau$.

Designate by $[\lambda_{ik}]$ the vectors the length of which one will carry out the translations of parallelohedra R_k into $R\,(k = 1, 2, \ldots, \tau)$.

By carrying out the translations of the face $P(\nu)$ the length of vectors $[\lambda_{ik}]\,(k = 1, 2, \ldots, \tau)$, one obtains the new faces

$$P'(\nu), P''(\nu), [\ldots,]P^{(\tau)}(\nu)$$

of the parallelohedron R.

Definition I. One will call congruent the faces of the parallelohedron R

$$P'(\nu), P''(\nu), \ldots, P^{(\tau)}(\nu)$$

in ν dimensions $(\nu = 0, 1, 2, \ldots, n-1)$.

15

Theorem. The number of parallelohedra of the set (R) which are contiguous by the same face in ν dimensions can not be less than $n+1-\nu\,(\nu=0,1,2,\ldots,n-1)$.

Suppose that the face $P(\nu)$ be determined within the parallelohedron R by the equation

$$a_{0r}+\sum a_{ir}x_i=0.\ (r=1,2,\ldots,\mu) \tag{1}$$

Designate by $R_1, R_2, \ldots, R_\mu$ the parallelohedra which are contiguous to R by the faces in $n-1$ dimensions defined by the equations (1). The face $P(\nu)$ will belong to all the parallelohedra $R_1, R_2, \ldots, R_\mu$, therefore

$$\tau \geq \mu.$$

As the face $P(\nu)$ is in ν dimensions, it is necessary that

$$\mu \geq n-\nu,$$

and as a result

$$\tau \geq n-\nu.$$

16

Definition II. One will call simple a face in ν dimensions which belong to only $n+1-\nu$ parallelohedra of the set (R).

Definition II. One will call primitive a parallelohedron, all the faces in different dimensions of which are simple.primitive parallelohedron

The primitive parallelohedra possess many important properties which simplify the study.

In the subsequent studies, one will study only the primitive parallelohedron and all the nonprimitive parallelohedra which can be considered as a boundary of primitive parallelohedra.

I am inclined to think that each primitive parallelohedron can be considered in this point of view, but I have not been successful in demonstrating this.

Section II

Fundamental properties of primitive parallelohedra

Definition of primitive parallelohedra.

17

We have called in Number 16 "primitive parallelohedron" all parallelohedron, all the faces in different dimensions of which are simple.

Theorem I. for a parallelohedron to be primitive it is necessary and sufficient that all the vertices be simple.

The theorem stated is evident by virtue of the definition established.

Theorem II. Two primitive parallelohedra belonging to the set (R) can be contiguous by only one face in $n-1$ dimensions.

Suppose that a face $P(\nu)$ in ν dimensions of a primitive parallelohedron R be determined with the aid of $n-\nu$ equations

$$a_{0r}+\sum a_{ir}x_i=0.\ (r=1,2,\ldots,n-\nu) \tag{1}$$

Designate by $R_1, R_2, \ldots, R_{n-\nu}$ the parallelohedra which are contiguous to the parallelohedron R by the faces in $n-1$ dimensions defined by the aid of equations (1). The face $P(\nu)$ will not belong to the parallelohedra $R_1, R_2, \ldots, R_{n-\nu}$ by virtue of the definition established, thus the theorem introduced is demonstrated.

Edges of primitive parallelohedra of the set (R)

18

Let (α_i) be a vertex of the primitive parallelohedron R determined by n equations

$$a_{0k}+\sum a_{ik}x_i=0.\ (k=1,2,\ldots,n) \tag{1}$$

Designate by $R_1, R_2, \ldots, R_n$ the parallelohedra contiguous to the parallelohedron R by the faces in $n-1$ dimensions determined with the help of equation (1).

By virtue of the definition established, the vertex (α_i) will not belong to the parallelohedra $R_1, R_2, \ldots, R_n$ of the set (R).

Determine n numbers $P_{1k}, P_{2k}, \ldots, P_{nk}$ with the help of equations

$$\sum a_{ir}P_{ik}=0.\ (r=1,2,\ldots,n; r\neq k; k=1,2,\ldots,n) \tag{2}$$

The equations (2) do not define the number $P_{1k}, P_{2k}, \ldots, P_{nk}$ to a common factor. Attach to the equations (2) a condition

$$\sum a_{ik}p_{ik}>0\ (k=1,2,\ldots,n) \tag{3}$$

and consider a vector g_k determined with the help of equalities

$$x_i=\alpha_i+p_{ik}\rho \ \text{ where } \ \rho\geq 0.$$

By attributing to the parameter ρ positive values sufficiently small, one will determine, by (3), the points of the vector g_k belonging to R. By putting

$$\alpha_{ik}=\alpha_i+p_{ik}\rho_k,$$

one will determine a vertex (α_{ik}) of the parallelohedron R adjacent to the vertex (α_i) by an edge $P_k(1)$ of the parallelohedron R $(k=1,2,\ldots,n)$. One will characterise the edge $P_k(1)$ by the symbol $[\alpha_i, \alpha_{ik}]$.

Observe that all the points of the edge $P_k(1)$ verifies $n-1$ equations

$$a_{0r}+\sum a_{ir}x_i=0.\quad (r=1,2,\ldots,n; r\neq k)$$

It follows that the edge $P_k(1)$ belongs to the parallelohedra

$$R, R_1, \dots, R_{k-1}, R_{k+1}, \dots, R_n \quad (k = 1, 2, \dots, n)$$

and by virtue of the definition established, does not belong to any other parallelohedron of the set (R).

One concludes that the parallelohedra

$$R_1, R_2, \dots, R_n$$

are contiguous by an edge too. By designating this edge by $P_0(1)$, one will determine it with the symbol $[\alpha_i, \alpha_{i0}]$ by putting

$$\alpha_{i0} = \alpha_i + p_{i0}\rho_0.$$

We have arrived at the following result:

There exist $n+1$ edges of parallelohedra of the set (R), contiguous by one common vertex of these parallelohedra.

Observe that $n-1$ edges

$$P_1(1), \dots, P_{k-1}(1), P_{k+1}(1), \dots, P_k(1) \quad (k = 1, 2, \dots, n)$$

define a face in $n-1$ dimensions which is common to the parallelohedra R and R_k $(k = 1, 2, \dots, n)$. Two parallelohedra R_k and R_h $(k = 1, 2, \dots, n; h = 1, 2, \dots, n)$ are contiguous by a face in $n-1$ dimensions which is defined by $n-1$ edges

$$P_r(1). \quad (r = 0, 1, 2, \dots, n; r \neq k, r \neq h)$$

Canonical form of equations which define a vertex of a primitive parallelohedron.

19 By conserving the previous notations, one can determine the vertex (α_i) within the parallelohedron R with the help of equations

$$u_k(a_{0k} + \sum a_{ik}x_i) = 0, \ (k = 1, 2, \dots, n) \tag{1}$$

$u_1, u_2, \dots, u_n$ being positive arbitrary parameters. One will say that the equation

$$-u_k(a_{0k} + \sum a_{ik}x_i) = 0 \ \text{ where } \ u_k > 0$$

does not define within the parallelohedron R a face in $n-1$ dimensions because the inequality

$$-u_k(a_{0k} + \sum a_{ik}x_i) \geq 0$$

will not satisfy all the points of the parallelohedron R.

Theorem. One can determine the positive values of parameters $u_1, u_2, \dots, u_n$ to a common factor, such that by putting

$$a'_{0k} = u_k a_{0k}, a'_{ik} = u_k a_{ik}, \ (i = 1, 2, \dots, n; k = 1, 2, \dots, n)$$

one will define the vertex (α_i) within the parallelohedron R by the equations

$$\sum a'_{ik}(x_i - \alpha_i) = 0, \ (k = 1, 2, \dots, n) \tag{2}$$

and one will define the vertex (α_i) within the parallelohedron R_k $(k = 1, 2, \dots, n)$ by the equations

$$\left\{ \begin{array}{l} \sum (a'_{ih} - a'_{ik})(x_i - \alpha_i) = 0, \ (h = 1, 2, \dots, n; h \neq k) \\ -\sum a'_{ik}(x_i - \alpha_i) = 0. \ (k = 1, 2, \dots, n) \end{array} \right. \tag{3}$$

Take an arbitrary positive parameter δ and determine the parameters $u_1, u_2, \dots, u_n$ after the equations

$$u_k \sum a_{ik}(\alpha_i - \alpha_{i0}) = \delta. \ (k = 1, 2, \dots, n) \tag{4}$$

I argue that the values $u_1, u_2, \dots, u_n$ obtained satisfy the conditions of the theory stated.

To demonstrate this, observe in the first place that the equations (4) define the positive values of $u_1, u_2, \dots, u_n$. In effect, we have seen in Number 18 that the edge $P_0(1)$ defined by the equalities

$$x_i = \alpha + u(\alpha_{i0} - \alpha_i) \ \text{ where } \ 0 \leq u \leq 1 \tag{5}$$

does not belong to the parallelohedra $R_1, R_2, \dots, R_n$. One concludes that by attributing to the parameter u any negative values sufficiently small, one will determine by the equality (5) a point which will be interior to the parallelohedron R. It follows that

$$\sum a_{ik}(\alpha_i - \alpha_{i0}) > 0, \quad (k = 1, 2, \dots, n)$$

and the equations (4) give

$$u_k > 0. \quad (k = 1, 2, \dots, n)$$

This established, designate by

$$\sum a^{(k)}_{ir}(x_i - \alpha_i) = 0 \ (r = 1, 2, \dots, n) \tag{6}$$

the equations which define the vertex (α_i) in the parallelohedron R_k $(k = 1, 2, \dots, n)$

Observe that n edges $P_r(1)$ $(r = 0, 1, 2, \ldots, n, r \neq k)$ are contiguous by the vertex (α_i) in the parallelohedron R_k. Each equation (6) will be verified by $n - 1$ edges. One can thus put

$$\begin{cases} \sum a^{(k)}_{ik}(\alpha_{ir} - \alpha_i) = 0, \ (r = 1, 2, \ldots, n; r \neq k) \\ \sum a^{(k)}_{ik}(\alpha_{i0} - \alpha_i) > 0 \end{cases} \tag{7}$$

and

$$\begin{cases} \alpha a^{(k)}_{ih}(\alpha_{ir} - \alpha_i) = 0, \ (r = 0, 1, 2, \ldots, n; r \neq k, r \neq h) \\ \sum a^{(k)}_{ih}(\alpha_{ih} - \alpha_i) > 0. \ (h = 1, 2, \ldots, n; h \neq k) \end{cases} \tag{8}$$

The conditions established define the coefficients of equations (6) to a common positive factor, which can be arbitrarily chosen.

Observe that the coefficients of equations (1), which define the vertex (α_i) in the parallelohedron R are also determined to a common positive factor and satisfy the conditions

$$\begin{cases} \sum a_{ik}(\alpha_{ir} - \alpha_i) = 0, \ (r = 1, 2, \ldots, n; r \neq k; k = 1, 2, \ldots, n) \\ \sum a_{ik}(\alpha_{ik} - \alpha_i) > 0 \end{cases} \tag{9}$$

The equalities (4), (7), (8) and (9), one takes

$$\left.\begin{aligned} a^{(k)}_{ik} &= -\delta_k u_k a_{ik}, \\ a^{(k)}_{ih} &= \delta_h(u_h a_{ih} - u_k a_{ik}), \end{aligned}\right\} (i = 1, 2, \ldots, n; h = 1, 2, \ldots, n; h \neq k)$$

where $\delta_1, \delta_2, \ldots, \delta_n$ are positive factors. One can put

$$\delta_1 = 1, \delta_2 = 1, \ldots, \delta_n = 1,$$

and the equations (6) become

$$\sum(u_h a_{ih} - u_k a_{ik})(x_i - \alpha_i) = 0, \ (h = 1, 2, \ldots, n; h \neq k)$$

$$-\sum u_k a_{ik}(x_i - \alpha_i) = 0.$$

The theorem introduced is thus demonstrated.

One will say that the equations (2) and (3) which define the vertex (α_i) in the contiguous parallelohedron $R, R_1, \ldots, R_n$ are presented in the canonical form.

We have seen in Number 18 that the parallelohedra R_k and R_h $(k = 1, 2, \ldots, n; h = 1, 2, \ldots, n)$ are contiguous by a face in $n-1$ dimensions. As this face is characterised by the edges $P_r(1)$ $(r = 0, 1, 2, \ldots, n; r \neq k; r \neq h)$, one will determine it in the parallelohedron R_h by the canonical equation

$$\sum(a'_{ik} - a'_{ih})(x_i - \alpha_i) = 0.$$

20 *Canonical form of inequalities which define a positive parallelohedron.*

Suppose that a primitive parallelohedron R is determined with the help of independent inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Bu designating with $u_1, u_2, \ldots, u_\sigma$ of arbitrary positive parameters, one will determine the parallelohedron R with the help of independent inequalities

$$u_k(a_{0k} + \sum a_{ik} x_i) \geq 0. \ (k = 1, 2, \ldots, \sigma) \tag{1}$$

We will see how all the problem of the study of primitive parallelohedra comes down to the appropriate choice of parameters $u_1, u_2, \ldots, u_\sigma$.

Fundamental Theorem. One can determine the positive values of parameters $u_1, u_2, \ldots, u_\sigma$ to a common factor, such that by putting

$$a'_{0k} = u_k a_{0k}, a'_{ik} - u_k a_{ik}, \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, \sigma)$$

one will determine the parallelohedron R with the help of inequalities

$$a'_{0k} + \sum a'_{ik} x_i \geq 0 \ (k = 1, 2, \ldots, \sigma) \tag{2}$$

which enjoy the following property: all the vertices of the parallelohedron R will be determined by the equations presented in the canonical form.

One will call the inequalities (2) canonical.

By conserving the previous notations, suppose that one had chosen the parameters $u_1, u_2, \ldots, u_n$ in such a manner that the vertex (α_i) is determined by the canonical equations

$$a'_{0k} + \sum a'_{ik} x_i = 0. \quad (k = 1, 2, \ldots, n)$$

Examine the equations which define a vertex (α_{ik}) $(k = 1, 2, \dots, n)$ of the parallelohedron R adjacent to the vertex (α_i) by the edge $P_k(1)$.

The vertex (α_{ik}) satisfies $n-1$ equations

$$a'_{0h} + \sum a'_{ih} x_i = 0. \; (h = 1, 2, \dots, n; h \neq k) \tag{3}$$

Designate by

$$b_{0k} + \sum b_{ik} x_i = 0 \tag{4}$$

the n^{th} equation which defines the vertex (α_{ik}).

Determine the positive parameters v_h $(h = 1, 2, \dots, n, h \neq k)$ and v_k corresponding to the equations (3) and (4), which reduces to these equations in the canonical form:

$$v_h(a'_{0h} + \sum a'_{ih} x_i) = 0 \quad (h = 1, 2, \dots, n; h \neq k)$$

and

$$v_k(b_{0k} + \sum b_{ik} x_i) = 0.$$

I argue that one can put

$$v_h = 1. \quad (h = 1, 2, \dots, n; h \neq k)$$

To demonstrate this, examine the canonical equation which defines in the parallelohedron R_h $(h = 1, 2, \dots, n; h \neq k)$ a face in $n-1$ dimensions common to the parallelohedra R_r and R_h $(r = 1, 2, \dots, n; r \neq h, r \neq, k)$.

By virtue of the theorem of Number 19, this face will be determined within R_h by the canonical equation

$$\sum (a'_{ir} - a'_{ih})(x_i - \alpha_i) = 0.$$

Besides, this same face will be determined in R_h, by virtue of the supposition made, by the canonical equation

$$\sum (v_r a'_{ir} - v_h a'_{ih})(x_i - \alpha_i) = 0.$$

It results in that

$$v_r a'_{ir} - v_h a'_{ih} = \delta(a'_{ir} - a'_{ih}), \quad (i = 1, 2, \dots, n)$$

and so

$$v_r = \delta, v_h = \delta,$$

thus

$$v_r = v_h. \quad (r = 1, 2, \dots, n; r \neq k; r \neq h)$$

As the parameters v_h $(h = 1, 2, \dots, n)$ are defined to a factor, one can put

$$v_h = 1, \quad (h = 1, 2, \dots, n; h \neq k)$$

and it only remains to determine the parameter v_k in order to define the vertex (α_{ik}) by the canonical equations.

21

By applying the procedure explained to all the vertices of the parallelohedron R adjacent to the vertex (α_i) and so on, one will successively determine the values of various parameters corresponding to all the inequalities (1).

It can turn out that one determines for one inequality the value of the corresponding parameter in various manners. I argue that all these values of the same parameter coincide.

The problem posed is extremely difficult. It is within this group of studies explained that is manifested their true geometrical characteristic, and one does not manage to master the difficulties which arise as a result with the help of geometrical methods.

Set of simplexes corresponding to the various vertices of a primitive parallelohedron.

22

We have seen in Number 4 that the various vertices of a parallelohedron R $(\alpha_{i1}), (\alpha_{i2}), \dots, (\alpha_{is})$ correspond to the domains

$$A_1, A_2, \dots, A_s \tag{1}$$

which uniformly fill the space in n dimensions.

By conserving the previous notations, examine a domain A which corresponds to the vertex (α_i) of the parallelohedron R.

The domain A is composed of points determined by the equalities

$$x_i = \sum_{k=1}^{n} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \; (k = 1, 2, \dots, n) \tag{2}$$

The domain A possesses n faces in $n-1$ dimensions which correspond to n edges $P_k(1), (k = 1, 2, \dots, n)$ contiguous by the vertex (α_i).

One will call the domain A simple.

Extract from the domain A a simplex L by the solution with the help of equalities

$$x_i = \sum_{k=1}^{n} \vartheta_k u_k a_{ik} \text{ where } \sum \vartheta_k \leq 1 \text{ and } \vartheta_k \geq 0. \; (k = 1, 2, \dots, n)$$

The simplex L possesses $n+1$ faces in $n-1$ dimensions which are opposite to $n+1$ vertices

$$(0), (u_1 a_{i1}), (u_2 a_{i2}), \ldots, (u_n a_{in}),$$

Examine the face of the simplex L which is opposite to the vertex (0). One can present the equation which defines this face in the form

$$1 - \sum p_i x_i = 0. \tag{3}$$

It follows that one will have an inequality

$$1 - \sum p_i x_i > 0$$

for any point of L which does not belong to the face examined.

As the vertices of L: $(u_k a_{ik})(k = 1, 2, \ldots, n)$ satisfy the equation (3), one has

$$\sum p_i a_{ik} = \frac{1}{u_k}, \ (k = 1, 2, \ldots, n) \tag{4}$$

thus

$$\sum p_i a_{ik} > 0. \quad (k = 1, 2, \ldots, n).$$

By virtue of (2), one obtains the inequality

$$\sum p_i x_i > 0$$

which holds for any point (x_i) of the domain A, the vertex (0) being excluded.

23 Examine in the same manner n domains $A_1, A_2, \ldots, A_n$ which are contiguous to the domain A by faces in $n-1$ dimensions.

One will take from the simple domain $A_k a, (k = 1, 2, \ldots, n)$ defined by the equalities

$$x_i = \sum \rho_h a_{ih} + \rho_k b_{ik} \text{ where } \rho_k \geq 0 \text{ and } \rho_h \geq 0, (h = 1, 2, \ldots, n; h \neq k)$$

a simplex L_k composed of points

$$x_i = \sum \vartheta_h u_h a_{ih} + \vartheta_k v_k b_{ik} \text{ where } \sum \vartheta_h + \vartheta_k \leq 1, \vartheta_k \geq 0, \vartheta_h \geq 0 \quad (h = 1, 2, \ldots, n; h \neq k)$$

Designate by

$$1 - \sum p_{ik} x_i = 0$$

the equation of the face of the simplex L_k which is opposite to the vertex (0).

One will have the equalities

$$\sum p_{ik} a_{ih} = \frac{1}{u_b} \quad (h = 1, 2, \ldots, n; h \neq k)$$

and

$$\sum p_{ik} b_{ih} = \frac{1}{u_k}.$$

By virtue of equalities (4), one obtains

$$\sum p_{ik} a_{ih} = \sum p_i a_{ih}. \quad (h = 1, 2, \ldots, n; h \neq k)$$

It follows that

$$\sum p_{ik} x_i = \sum p_i x_i, \tag{5}$$

for any point (x_i) belonging to the face common to domains A and A_k.

By applying the procedure explained to the domains which are contiguous to the domains $A_1, A_2, \ldots, A_n$ and so on, one will extract from any domain of the set (1) a corresponding simplex.

It can turn out that one extracts from the same domain the corresponding simplex by various manners. I argue that all these simplexes coincide.

It is clear that the problem stated does not differ from a formulation of the problem put forward in Number 21.

We shall show a new formulation of this problem.

On a function defined by the set of simplexes corresponding to the various vertices of a primitive parallelohedron.

24 Introduce within our study a function $P(x_1, x_2, \ldots, x_n)$ of variables $x_1, x_2, \ldots, x_n$ by defining as follows.

1. One will determine the function $P(x_1, x_2, \ldots, x_n)$ in the domain A by the formula

$$P_{(A)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i.$$

2. In the domains A_k $(k = 1, 2, \ldots, n)$ contiguous to the domain A by the faces in $n - 1$ dimensions, one will determine the function $P(x_1, x_2, \ldots, x_n)$ by the formula

$$P_{(A_k)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} P_{ik}x_i. \quad (k = 1, 2, \ldots, n)$$

Let

$$A, A', A'', \ldots, A^{(m)} \tag{1}$$

be a series of domains which are successively contiguous by the faces in $n-1$ dimensions. One will successively take from these domains the following simplexes.

$$L, L', L'', \ldots, L^{(m)}$$

and one will determine the corresponding function.

$$\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i' x_i, \sum_{i=1}^{n} p_i'' x_i, \ldots, \sum_{i=1}^{n} p_i^{(m)} x_i.$$

One will define the function $P(x_1, x_2, \ldots, x_n)$ in the domains (1) by the formula

$$P_{(A^{(k)})}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} p_i^k x_i. \quad (k = 1, 2, \ldots, m)$$

Fundamental Theorem. The function $P(x_1, x_2, \ldots, x_n)$ defined by the conditions 1, 2, and 3 is continuous and uniform in all the space in n dimensions.

Observe that the fundamental introduced only give as a new formulation of the fundamental theorem of Number 20.

Take an arbitrary closed contour C. By traversing the contour C, one can determine a series of domains successively contiguous by faces in $n - 1$ dimensions in which belong the points of the contour C:

$$A^{(0)}, A', A'', \ldots, A^{(m)}, A^{(0)}, A', \ldots$$

To demonstrate this, take a point (ξ_{i0}) of the contour C and designate by C_0 a curve which is being traversed within a domain $A^{(0)}$ leaving from the initial point (ξ_{i0}). Suppose that the curve C_0 does not coincide with the contour C and designate by (ξ_{i1}) the final point of the curve C_0.

Suppose that on leaving the point (ξ_i) one got out of the domain $A^{(0)}$ and that one entered inside the domain A'. Designate by C_1 a group of contour C which one has traversed in the domain A' when leaving the point (ξ_{i1}) and so on and so forth. Suppose that one had divided with the help of the procedure described the contour C into groups

$$C_0, C_1, \ldots, C_m, C_0$$

which belong to the domains

$$A^{(0)}, A', A'', \ldots, A^{(m)}, A^{(0)}. \tag{2}$$

It can turn out that two adjacent domains of this series $A^{(k)}$ and $A^{(k+1)}$ are not contiguous by a face in $n - 1$ dimensions. One inserts in this case between the domains $A^{(k)}$ and $A^{(k+1)}$ new domains of the solutions as follows:

A point $(\xi_{i,k+1})$ which is the final point of the curve C_k and which gives the initial point of the curve C_{k+1} belongs, by virtue of the supposition made, to the domains $A^{(k)}$ and $A^{(k+1)}$. One concludes that the point $(\xi_{i,k+1})$ is interior to a face $A^{(k)}(\nu)$ in ν dimensions which is common to the domains $A^{(k)}$ and $A^{(k+1)}$.

One can determine a parallelepiped K with the help of equalities

$$x_i = \xi_{i,k+1} + u_i \ \text{ where } \ |u_i| \leq \epsilon, \ (i = 1, 2, \ldots, n)$$

of manner such that the points of the parallelepiped D do not belong to the domains of the series (1) (Number 22) which are contiguous by the face $A^{(k)}(\nu)$.

Take with the parallelepiped K two points (x_{ik}) and $x_{i,k+1}$ which ae interior to the domain $A^{(k)}$ and $A^{(k+1)}$ and take within the parallelepiped K a curve $C^{(k)}$ which joins the points (x_{ik}) and $(x_{i,k+1})$. One can choose this curve in such a manner that it does not pass beyond any face of domains (1) (Number 22) of which the number of dimension s is less than $n - 1$.

Suppose that the curve $C^{(k)}$ traverse the domain

$$A^{(k)}, A_1^{(k)}, \ldots, A_\mu^{(k)}, A^{(k+1)}.$$

By virtue of the supposition made, the domains obtained are successively contiguous by the faces in $n - 1$ dimensions. All these domains are contiguous pairwisely by the face $A^{(k)}(\nu)$.

In the same manner, one will examine all the pairs of adjacent domains of the series (2) and one will form the series

$$A^{(0)}, A', \ldots, A^{(m)}, A^{(0)}$$

of domains successively contiguous by the faces in $n - 1$ dimensions to which belong all the points of the closed contour C given.

25 This established, observe that the fundamental theorem introduced is true in the case where all the domains (2) are contiguous in at least one edge.

In effect, suppose that one had successively taken away from the domains (2) the simplexes

$$L^{(0)}L', L'', \dots, L^{(m)}, L^{(m+1)} \tag{3}$$

I argue that the simplex $L^{(m+1)}$ taken from the domain $A^{(0)}$ coincide with the simplex $L^{(0)}$. To demonstrate this, designate by

$$\sum p_i^{(m+1)} x_i = \delta \sum p_i^{(0)} x_i\,. \tag{4}$$

By virtue of the supposition made, the domains (2) are contiguous by at least one edge. Let (a_i) be a point of this edge.

As the domains $A^{(0)}$ and A' are contiguous by a face in $n-1$ dimensions, one will have, as we have seen this in Number 23, an equality

$$\sum p_i^{(0)} x_i = \sum p'_i x_i \tag{5}$$

which holds for any point (x_i) of thee face common to the domains $A^{(0)}$ and A'.

By making $x_i = a_i$, one obtains

$$\sum p^{(0)} a_i = \sum p'_i a_i$$

In the same manner, one will obtain

$$\sum p_i^{(0)} a_i = \sum p'_i a_i = \dots \sum p_i^{(m)} a_i = \sum p_i^{(m+1)} a_i\,.$$

On the other hand, the identity (4) gives

$$\sum p_i^{(0)} a_i = \delta \sum p_i^{(m+1)} a_i\,,$$

and as $\sum p_i^{(0)} a_i > 0$, then $\delta = 1$, therefore

$$\sum p_i^{(m+1)} x_i = \sum p_i^{(0)} x_i$$

and the two simplexes $L^{(0)}$ and $L^{(m+1)}$ coincide.

By virtue of the definition established, one will determine the function $P(x_1, \dots, x_n)$ in the domain $A^{(0)}$ by the formula

$$p_{(A^0)}(x_1, x_2, \dots, x_n) = \sum p_i^{(}0) x_i$$

by leaving the domain $A^{(0)}$ and by returning to within that domain after having traversed the path C.

26 We will see that the general case can be brought back to the case examined. To this effect, suppose the projection of any one contour C evaluated in relation to surface S is determined by the equation

$$\sum x_i^2 = 1.$$

By putting

$$x'_i = \frac{x_i}{\sqrt{\sum x_i^2}}, \quad (i = 1, 2, \dots, n) \tag{6}$$

one will call the point (x'_i) the projection of the point (x_i) within the surface S.

Designate by C' a projection of this contour C.

Suppose that by traversing the contour C', one returns to the initial point (ξ'_i) with the same solution of the function $P(x_1, \dots, x_n)$ of which when leaving that point. I argue that one will return to the corresponding point ξ_i of the contour C with the same solution of the function $P(x_1, \dots, x_n)$.

To demonstrate this, it suffices to observe that the points (ξ_i) and (ξ'_i), by virtue of equalities (6), belong to the same domains of the series (2).

One concludes that it suffices to examine the different closed contours belonging to the surface S.

27 Introduce in our study a function $d(x_i, x'_i)$ being defined by the formula

$$d(x_i, x'_i) = \sqrt{\sum (x'_i - x_i)^2}.$$

One will call distance between two points (x_i) and (x'_i) the corresponding value of the function $d(x_i, x'_i)$.

Lemma. One can determine a positive parameter δ satisfying the following condition: every closed contour C belonging to the surface S will be situated in the domains which are contiguous by at least one edge, if the distance of all the point of the contour C, each of all to the rest, do not exceed the limit δ.

Let (ξ_i) be a point of the contour C belonging to the domain A. Put

$$\xi_i = \sum_{k=1}^{n} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \dots, n)$$

By virtue of the equation

$$\sum \xi_i^2 = 1,$$

the sum $\sum_{k=1}^{n} \rho_k$ is not less than a positive fixed limit.

$$\sum_{k=1}^{n} \rho_k \geq \tau. \tag{7}$$

Suppose that the contour C is not situated entirely within the domain A.
Let (ξ_i') be a point of C which does not belong to the domain A. By putting

$$\xi_i' = \sum_{k=1}^{n} \rho_k' a_{ik}, \tag{8}$$

one will have among the numbers $\rho_1', \rho_2', \dots, \rho_n'$ at least one negtive number.
Suppose, to fix an ideas, that

$$\rho_1' \geq 0, \rho_2' \geq 0, \dots, \rho_\mu' \geq 0 \tag{9}$$

and that

$$\rho_{\mu+1}' < 0, \rho_{\mu+2}' < 0, \dots, \rho_n' < 0. \tag{10}$$

After the supposition made, one has the inequality

$$d(\xi_i, \xi_i') \geq \delta.$$

One can choose the parameter δ, of such a manner that one had the inequalities

$$|\rho_k' - \rho_k| < \epsilon, \ (k = 1, 2, \dots, n) \tag{11}$$

ϵ being a positive parameter also small as one would wish.
By (10), one obtains

$$0 \leq \rho_k < \epsilon, -\epsilon < \rho_k' < 0. \ (k = \mu+1; \mu+2, \dots, n) \tag{12}$$

Choose among the numbers $\rho_1, \rho_2, \dots, \rho_n$ the one which is the largest. By virtue of the inequality (7), this number can not be less than $\frac{\tau}{n}$. By supposing that

$$\epsilon < \frac{\tau}{n},$$

one will find the number looked for among the numbers $\rho_1, \rho_2, \dots, \rho_\mu$. Suppose, to fix the ideas, that

$$\rho_1 > \frac{\tau}{n}.$$

the inequality (11) gives

$$\rho_1' > \frac{\tau}{n} - \epsilon. \tag{13}$$

This posed, suppose that the point (ξ_i') belonged to the domain A' and put

$$\xi_i' = \sum_{k=1}^{n} u_k a_{ik}' \text{ where } u_k \geq 0. \ (k = 1, 2, \dots, n) \tag{14}$$

Designate by (α_i) and α_i' two vertices of the parallelohedron R corresponding to domains A and A' by defining them by the equations

$$a_{0k} + \sum a_{ik} x_i = 0, \quad (k = 1, 2, \dots, n)$$

and by the equations

$$a_{0k}' + \sum a_{ik}' x_i = 0. \ (k = 1, 2, \dots, n) \tag{15}$$

By virtue of equality (8) and (14), one obtains an identity

$$\rho_0' + \sum_{k=1}^{n} \rho_k' (a_{0k} + \sum a_{ik} x_i) = \sum_{k=1}^{n} u_k (a_{0k}' + \sum a_{ik}' x_i). \tag{16}$$

By making in this identity $x_i' = \alpha_i$, one finds

$$\rho_0' = \sum u_k (a_{0k}' + \sum a_{0k}' \alpha_i) \geq 0. \tag{17}$$

By making in the identity (16) $x_i = \alpha_i'$, it will become

$$\rho_0' + \sum_{k=1}^{n} \rho_k' (a_{0k} + \sum a_{ik} \alpha_i') = 0. \tag{18}$$

Suppose that

$$a_{01} + \sum a_{i1} \alpha_i' > 0.$$

By virtue of (7), (12), (13) and (17), one will have

$$\rho_0' + \sum_{k=1}^{\mu} \rho_k'(a_{0k} + \sum a_{ik}\alpha_i') > (\frac{\tau}{n} - \epsilon)(a_{01} + \sum a_{i1}\alpha_i'),$$

$$\sum_{k=\mu+1}^{n} \rho_k'(a_{0k} + \sum a_{ik}\alpha_i') \geq -\epsilon \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha_i'),$$

and the equality (18) gives

$$\frac{\tau}{n}(a_{01} + \sum a_{i1}\alpha_i') < \epsilon \left[a_{01} + \sum a_{i1}\alpha_i' + \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha_i') \right]. \tag{19}$$

Designate

$$A = \frac{\tau}{n}(a_{01} + \sum a_{i1}\alpha_i') \text{ and } B = a_{01} + \sum a_{i1}\alpha_i' + \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha_i'),$$

one will have

$$A > 0 \text{ and } B > 0,$$

and as a result

$$\epsilon > \frac{A}{B}. \tag{20}$$

One could determine the ratio $\frac{A}{B}$ correspondent to the different vertices of the parallelohedron R. Designate by ω the smallest of these ratios which is not zero. The parameter ϵ being arbitrary, one can suppose that

$$\epsilon \leq \omega.$$

The inequality (20) becomes impossible, it is therefore necessary that $A = 0$ or [to put it] differently

$$a_{01} + \sum a_{i1}\alpha_i' = 0.$$

By virtue of the equality obtained, the coefficients of the equation

$$a_{01} + \sum a_{i1}x_i = 0$$

are proportional to those of an equation which is among the equations (15).

By putting

$$a_{01} + \sum a_{i1}x_i = u(a_{0h}' + \sum a_{ih}'x_i) \text{ where } u > 0,$$

one will have

$$a_{i1} = ua_{ih}'. \quad (i = 1, 2, \ldots, n)$$

We have arrived at the following result: all the domain traversed by the contour C examined are contiguous by at least one edge which is characterised by the point (a_{i1}).

28 We are now in the state of reaching the demonstration of the fundamental theorem announced.

Let C be any contour belonging to the surface S. Suppose that on leaving the point (ξ_i) one passes via the points $(\xi_i^{(0)})$, (ξ_i'), (ξ_i'') and one returns to the point (ξ_i).

The path around the contour C can be replaced by the paths $C^{(0)}$ and C'.

The contour $C^{(0)}$ will be composed of a group $(\xi_i) - (\xi_i^{(0)})$ of C, of the vector $[\xi_i^{(0)}, \xi_i'']$ and of a group $(\xi_i^{(0)}) - (\xi_i') - (\xi_i'')$ of the contour C and of the vector $[\xi_i'', \xi_i^{(0)}]$.

Suppose that by traversing the paths $C^{(0)}$ and C' one uniformly defined the function $P(x_1, x_2, \ldots, x_n)$. In this case the trajectory by the group $(\xi_i^{(0)}) - (\xi_i')$ of the contour C can be replace by the path the length of the vector $[\xi_i^{(0)}, \xi_i'']$.

By replacing the group $(\xi_i^{(0)}) - (\xi_i') - (\xi_i'')$ of the contour C by the vector $[\xi_i^{(0)}, \xi_i'']$, one will transform the contour C to $C^{(0)}$, thus, by traversing the contour C, one will return to the point (ξ_i), by virtue of suppositions made, with the same solution of the function $P(x_1, x_2, \ldots, x_n)$.

Two contours $C^{(0)}$ and C' can be examined in the same manner and so on.

Suppose that one had determined the contours

$$C_1, C_2, \ldots, C_m \tag{21}$$

which replace the path C. By supposing that the function $P(x_1, x_2, \ldots, x_n)$ be uniform the length of contour (21), one will demonstrate that it will be uniform the length of the contour C given.

This established, observe that we can always choose the contours (21), of such a manner that their contours satisfy the conditions of the lemma of the previous Number. In this case, any contour (21) will be situated within domains which are contiguous by at least one edge. We have seen in Number 25 that by traversing the same contours one will always return to the point of departure by the same solution of the function $P(x_1, x_2, \ldots, x_n)$ as while leaving this point. It is thus demonstrated that any closed contour C possesses the same property.

We have demonstrated that the function $P(x_1, x_2, \ldots, x_n)$ is uniformly defined in any domain of the set (1) (Number 22). It remains to demonstrate that the function $P(x_1, x_2, \ldots, x_n)$ is well defined in any point of the space in n dimensions.

Suppose that a point ξ_i belongs to two domains A and $A^{(0)}$.

I argue that the function $P(x_1, x_2, \ldots, x_n)$ for the point ξ_i will have one same value in the domain A and in the domain $A^{(0)}$.

To demonstrate this, one will form a series of domains

$$A, A', \ldots, A^{(m)}, A^{(0)}$$

which are successively contiguous by faces in $n-1$ dimensions and in which belongs the point ξ_i.

As the point ξ_i belongs to the face common to domains A and A', one will have ,by virtue of the formula (5) of Number 23,

$$P_{(A)}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A')}(\xi_1, \xi_2, \ldots, \xi_n).$$

In the same manner, one obtains

$$P_{(A')}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A'')}(\xi_1, \xi_2, \ldots, \xi_n),$$

$$\ldots$$

$$P_{(A^{(m)})}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A^{(0)})}(\xi_1, \xi_2, \ldots, \xi_n).$$

It results in that

$$P_{(A)}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A^{(0)})}(\xi_1, \xi_2, \ldots, \xi_n).$$

The fundamental theorem announced is thus demonstrated.

Canonical form of inequalities which define the set (R) of primitive parallelohedron S.

29

Choose within the set (R) of primitive parallelohedra any parallelohedron R_0 . Suppose that the parallelohedron R_0 is determined with the help of canonical inequalities

$$a_{0k} + \sum a_{ik}x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Observe that we can replace these inequalities by the following canonical inequalities:

$$u(a_{0k} + \sum a_{ik}x_i) \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

u being a positive arbitrary parameter.

Designate by R_k $(k = 1, 2, \ldots, \sigma)$ the parallelohedron which is contiguous to the parallelohedron R_0 by the face determined within R_0 by the equation

$$a_{0k} + \sum a_{ik}x_i = 0 \tag{1}$$

and suppose that the vector $[\lambda_{ik}]$ defined a translation of the parallelohedron R_k to R_0.

It follows that the parallelohedron $R_k$will be determined by the canonical inequalities

$$a_{0h} + \sum a_{ih}(x_i + \lambda_{ik}) \geq 0, \quad (h = 1, 2, \ldots, \sigma)$$

or by the canonical inequalities

$$u_k[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})] \geq 0, \ (h = 1, 2, \ldots, \sigma) \tag{2}$$

u_k being an arbitrary positive parameter.

The face P_k in $n-1$ dimensions common to the parallelohedra R and R_k is defined in the parallelohedron R_0 by the equation (1). Within the parallelohedron R_k, the face P_k will be determined by an equation in which the coefficients are proportional to those of the equation

$$-a_{0k} - \sum a_{ik}x_i = 0.$$

One can choose the positive parameter u_k, of a manner such that one had the identity

$$-a_{0k} - \sum a_{ik}x_i = u_k(a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})).$$

In this case, the inequality

$$-a_{0k} - \sum a_{ik}x_i \geq 0$$

is found among the inequalities (2) which define the parallelohedron R_k.

One will say that these inequalities are represented in the canonical form.

30

Observe an important property of canonical inequalities which define the parallelohedra $R_0, R_1, R_2, \ldots, R_\sigma$.

Let (α_i) be a vertex of the parallelohedron R_0 determined by the canonical equations

$$a_{0k} + \sum a_{ik}x_i = 0. \ (k = 1, 2, \ldots, n) \tag{3}$$

Examine the canonical equations which define the vertex (α_i) in the parallelohedron R_k $(k = 1, 2, \ldots, n)$.

The equations (3) being canonical, one will determine the vertex (α_i) within the parallelohedron R_k, by virtue of the theorem of Number 19, by the equation

$$\sum(a_{ih}-a_{ik})(x_i-\alpha_i)=0, \quad (h=1,2,\dots,n,h\neq k)$$

$$-\sum a_{ik}(x_i-\alpha_i)=0 \quad (k=1,2,\dots,n)$$

By virtue of the supposition made, the inequality

$$-\sum a_{ik}(x_i-\alpha_i)\geq 0$$

exists among the canonical inequalities (2) which define the parallelohedron R_k, which results in that the inequalities

$$\sum(a_{ih}-a_{ik})(x_i-\alpha_i)\geq 0, \quad (h=1,2,\dots,n,h\neq k)$$

also exist among the canonical inequalities (2).

One concludes that the canonical equation

$$\sum(a_{ih}-a_{ik})(x_i-\alpha_i)=0$$

define in the parallelohedron R_k a face in $n-1$ dimensions which is common to the parallelohedra R_k and R_h. the same face will be determined in the parallelohedron R_h by a canonical equation

$$\sum(a_{ik}-a_{ih})(x_i-\alpha_i)=0.$$

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by applying the procedure explained, one can determine the canonical inequalities which define the parallelohedra contiguous to the parallelohedra $R_1,R_2,\dots,R_\sigma$ and so on.

For every parallelohedron R of the set (R), one can form a series of parallelohedra

$$R_0,R',R'',\dots,R^{(m)},R$$

which are successively contiguous. One will determine successively the canonical inequalities that define the parallelohedra of this series.

One could arrive at the parallelohedron R by other ways and determine the canonical inequalities which define the parallelohedron R in various manners.

We shall se that the canonical inequalities which define a parallelohedron of the set (R) do not depend on the path by which one arrives at the parallelohedron (R) leaving from the principal parallelohedron R_0.

Generatrix function of the set (R) *of primitive parallelohedra.*

32

Consider a set (R) of primitive parallelohedra. Suppose any parallelohedron R of the set (R) be characterised by a vector $[\lambda_i]$ which defines a translation of parallelohedra R to a principal parallelohedron R_0.

Designate by G the group of vectors $[\lambda_i]$ which correspond to the different parallelohedra of the set (R).

Introduce in our study a function

$$V(x_1,x_2,\dots,x_n,\lambda_1,\lambda_2,\dots,\lambda_n)$$

of variables $x_1,x_2,\dots,x_n$ and parameters $\lambda_1,\lambda_2,\dots,\lambda_n$ by defining it within the space in n dimensions and for the group G such that:

1. Within the principal parallelohedron R_0, one will write

$$V(x_1,x_2,\dots,x_n,0,0,\dots,0,)=0.$$

2. Within the parallelohedron R_k which is contiguous to R_0, one will write

$$V(x_1,x_2,\dots,x_n,\lambda_{1k},\lambda_{2k},\dots,\lambda_{nk}=a_{0k}+\sum a_{ik}x_i, \ (k=1,2,\dots,\sigma)$$

providing that in the parallelohedron R_0 the canonical equation

$$a_{0k}+\sum a_{ik}x_i=0$$

had the face in $n-1$ dimensions common to the parallelohedra R_0 and R_k.

3. By supposing that the parallelohedra R and R' characterised by the vectors $[\lambda_i]$ and λ_i' are contiguous by a face in $n-1$ dimensions which is defined within R by a canonical equation

$$a_0+\sum a_ix_i=0,$$

one will write

$$V(x_1,x_2,\dots,x_n,\lambda_1,',\lambda_2',\dots,\lambda_n')=$$

$$V(x_1,x_2,\dots,x_n,\lambda_1,\lambda_2,\dots,\lambda_n)+a_0+\sum a_ix_i.$$

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Let R be any parallelohedron of the set (R) characterised by a vector $[\lambda_i]$. One will form a series of parallelohedra.

$$R_0,R',\dots,R^{(m)},R$$

which are successively contiguous by faces in $n-1$ dimensions. Designate by

$$a_0^{(0)} + \sum a_i^{(0)} x_i = 0$$

the equation of the faces common to the parallelohedra R_0 and R' and defined in R_0; designate by

$$a_0' + \sum a_i' x_i = 0$$

the equaion of the face common to the parallelohedra R' and R'' defined in R' and so on.

By applying the definition established, one will determine the function

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$$

by the formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i).$$

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Fundamental theorem. The function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is well defined for any vector $[\lambda_i]$ of the group G.

Suppose that one had formed a series of parallelohedra

$$R, R', R'', \ldots, R^{(m)}, R \tag{1}$$

which are successively contiguous. On leaving the parallelohedron R with any solution of the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$, one will return inside the parallelohedron R after having traversed the parallelohedra (1) with a solution of the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ which, by virtue of the definition established, is expressed by the vertex

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) + \sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i).$$

We shall demonstrate that one will always have

$$\sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i) = 0.$$

Examine, in the first place, the case where all the parallelohedra (1) are contiguous by at least one vertex (α_i). By virtue of Theorem II of Number 17, all the primitive parallelohedra (1) will be in this case contiguous one to one through faces in $n-1$ dimensions.

Designate by

$$a_{0k} + \sum a_{ik}(x_i - \alpha_i) = 0, \quad (k = 1, 2, \ldots, m)$$

the canonical equation of the face common to the parallelohedra $R^{(k)}$ and R $(k = 1, 2, \ldots, m)$ defined within the parallelohedron R.

We have seen in Number 30 that the canonical equation of the face common to the parallelohedra R' and R'' and defined within R' will be

$$\sum (a_{i2} - a_{i1})(x_i - \alpha_i) = 0$$

and so on and so forth. One obtains the formulae

$$\begin{aligned}
a_0^{(0)} + \sum a_i^{(0)} x_i &= \sum a_{i1}(x_i - \alpha_i), \\
a_0' + \sum a_i' x_i &= \sum (a_{i2} - a_{i1})(x_i - \alpha_i), \\
&\ldots \\
a_0^{(m-1)} + \sum a_i^{(m-1)} x_i &= \sum (a_{im} - a_{i,m-1})(x_i - \alpha_i), \\
a_0^{(m)} + \sum a_i^{(m)} x_i &= -\sum a_{im}(x_i - \alpha_i),
\end{aligned}$$

and it follows that

$$\sum_{k=0}^{m} (a_0^{(k)} + \sum a_i^{(k)} x_i) = 0.$$

35

We shall see that the general case can be brought back to the case examined.

Theorem. One can determine a positive parameter δ, in a manner that every closed contour C is found within the parallelohedra which are contiguous by at least one vertex, providing that the distance between any two points of the contour C does not exceed the limit δ.

Observe, in the first place, that the distance between the points (ξ_i) and (ξ'_i) belonging to the two parallelohedra which are not contiguous can not be less than a fixed limit. To demonstrate this, suppose that the point (ξ_i) belong to the parallelohedron R defined with the help of inequalities

$$a_{0k} + \sum a_{ik}x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by $R_1, R_2, \ldots, R_\sigma$ the parallelohedra which are contiguous to R and examine the set K of points belonging to the parallelohedra $R_1, R_2, \ldots, R_\sigma$.

Designate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

the vertices of parallelohedron R and designate by

$$(\alpha_{i1}^{(h)}), (\alpha_{i2}^{(h)}), \ldots, (\alpha_{is}^{(h)}), \quad (h = 1, 2, \ldots, \sigma)$$

the vertices of parallelohedron R_h $(h = 1, 2, \ldots, \sigma)$.

By virtue of the supposition made, one will have the inequalities

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0, \ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma).$$

Designate by ρ the smallest numerical value of vertices

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \quad (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma)$$

which does not become zero. By virtue of supposition made, one will have the inequality

$$\rho + a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0,$$

on condition that

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} < 0,$$

where $(k = 1, 2, \ldots, s,\ h = 1, 2, \ldots, \sigma)$.

This established, take any point (ξ'_i) which does not belong to the set K. Examine the points of a vector $[\xi_i, \xi'_i]$. By putting

$$x_i = \xi_i + u(\xi'_i - \xi_i) \text{ where } 0 \leq u \leq 1,$$

let us think the parameter u of a continuous manner within the interval $0 < u < 1$. One will determine a point

$$\xi_i^{(0)} = \xi_i + u_0(\xi'_i - \xi_i) \text{ where } 0 < u_0 < 1 \tag{2}$$

which belongs to the boundary of the set K, that is to say to a face in $n - 1$ dimensions of parallelohedra $R_1, R_2, \ldots, R_\sigma$ and which also belongs to another parallelohedron R'.

Suppose that the point $(\xi_i^{(0)})$ belongs to the parallelohedron R_h. The parallelohedra R_h and R' will be contiguous by a face in $n - 1$ dimensions.

Designate by

$$(\alpha_{i1}^{(h)}), (\alpha_{i2}^{(h)}), \ldots, (\alpha_{it}^{(h)}) \tag{3}$$

the vertices of parallelohedron R_h which belong to this face.

None of these vertices verifies the equation

$$a_{0h} + \sum a_{ih}x_i = 0$$

because otherwise the face examined would belong to two parallelohedron of the series $R, R_1, \ldots, R_\sigma$, which is contrary to the hypothesis.

Therefore one will have the inequalities

$$\rho + a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0. \quad (k = 1, 2, \ldots, t)$$

The point $(\xi_i^{(0)})$ belonging to the face of R_h, which is characterised by the vertices (3), probably determined by the equations

$$\xi_i^{(0)} = \sum_{k=1}^{k=t} \vartheta_k\alpha_{ik}^{(h)} \text{ where } \sum_{k=1}^{k=t} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0. (k = 1, 2, \ldots, t)$$

Of the previous inequalities, one draws

$$\rho + a_{0h} + \sum a_{ih}\xi_i^{(0)} \leq 0.$$

By observing that on the other hand one has

$$a_{0h} + \sum a_{ih}\xi_i \geq 0, \tag{4}$$

one finds, by (2),

$$\rho + a_{0h} + \sum a_{ih}\xi'_i < 0. \tag{5}$$

By virtue of inequalities (4) and (5), the distance $d(\xi_i, \xi_i')$ can not be smaller than a fixed limit d.

Solution of the centre of the primitive parallelohedra

36 This established, examine a contour C formed which the points had the mutual distance that does not surpass δ . By supposing that

$$\delta < d,$$

one will have a contour C which is situated within the contiguous parallelohedra two to two. [one to one]

Let ξ_i be any point of the contour C belonging to the parallelohedron R. Suppose that not all the points of contour C belong to R and designate by ξ_i' a point of contour C which does not belong to R. Put

$$\xi_i = \sum_{k=1}^{s} \vartheta_k \alpha_{ik} \text{ where } \sum \vartheta_k = 1 \text{ and } \vartheta_k \geq 0,\ (k = 1, 2, \ldots, s) \tag{6}$$

$$\xi_i' = \sum_{k=1}^{s} \vartheta_k' \alpha_{ik} \text{ where } \sum \vartheta_k' = 1 \tag{7}$$

As the point (ξ_i') does not belong to R, one will have among the numbers $\vartheta_1', \vartheta_2', \ldots, \vartheta_s'$ at least one number which will be negative. Suppose, to fix the ideas, that

$$\vartheta_1' \geq 0, \ldots, \vartheta_\mu' \geq 0 \text{ and } \vartheta_{\mu+1}' < 0, \ldots, \vartheta_s' < 0 \tag{8}$$

One can choose the parameter δ as small that one would have the inequalities

$$|\vartheta_k' - \vartheta_k| < \epsilon, \quad (k = 1, 2, \ldots, s) \tag{9}$$

ϵ being a positive parameter also as small as one would like. By virtue of (8), it will become

$$0 \leq \vartheta_k < \epsilon,\ -\epsilon < \vartheta_k' < 0. \quad (k = \mu + 1, \ldots, s) \tag{10}$$

Observe how the largest one among the numbers $\vartheta_1, \vartheta_2, \ldots, \vartheta_s$ can not be smaller than $\frac{1}{s}$ by (6) by supposing that

$$\epsilon < \frac{1}{s},$$

one wil find the required number among the number $\vartheta_1, \vartheta_2, \ldots, \vartheta_\mu$. Suppose, for fixing ideas, that

$$\vartheta_1 > \frac{1}{s}.$$

By virtue of (9), it will become

$$\vartheta_1' > \frac{1}{s} - \epsilon \tag{11}$$

We have demonstrated that the point (ξ_i') can not belong to these parallelohedra $R_1, R_2, \ldots, R_\sigma$ which are contiguous to R. By supposing that the point (ξ_i') belong to the parallelohedron R_h, one will have an equality

$$a_{0h} + \sum a_{ih} \xi_i' < 0 \tag{12}$$

Observing that by virtue of equations (7)

$$a_{0h} + \sum a_{ih} \xi_i' = \sum_{k=1}^{s} \vartheta_k' (a_{0h} + \sum a_{ih} \alpha_{ik}),$$

one obtains, because of (12),

$$\sum_{k=1}^{s} \vartheta_k' (a_{0h} + \sum a_{ih} \alpha_{ik}) < 0$$

Of this inequality one draws, by (10) and (11),

$$\frac{1}{2}(a_{0h} \sum a_{ih} \alpha_{i1}) - \epsilon \left[a_{0h} \sum a_{ih} \alpha_{i1} + \sum_{k=\mu+1}^{s} (a_{0h} + \sum a_{ih} \alpha_{ik}) \right] < 0\,.$$

By putting

$$A = \frac{1}{2}(a_{0h} \sum a_{ih} \alpha_{i1}) \text{ and } B = a_{0h} + \sum a_{ih} \alpha_{i1} + \sum_{k=\mu+1}^{s} (a_{0h} + \sum a_{ih} \alpha_{ik}),$$

suppose that $A > 0$; the previous inequality gives $B > 0$, thus

$$\epsilon > \frac{A}{B} \tag{13}$$

Observe that the numbers A and B do not change when one replace the parallelohedron by any parallelohedron of the set (R). One concludes that the ratio $\frac{A}{B}$ which does not vanish possess a positive minimum ω.

By supposing that

$$\epsilon < \omega,$$

the inequality (13) becomes impossible and it is necessary that $A = 0$ or otherwise

$$a_{0h} + \sum a_{ih}\alpha_{i1} = 0$$

We have arrived at the following result: all the parallelohedra within which is situated the contour examined C are contiguous by the vertex (α_{i1}).

With the help of the lemma of Number 35, one will easily demonstrate the fundamental theorem stated by repeating the reasoning explained in Number 28.

Fundamental properties of the generatrix function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$

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Theorem I. Suppose that two vectors $[\lambda_i]$ *and* $[\lambda_i^{(0)}]$ *characterise two parallelohedra* R *and* $R^{(0)}$ *of the set* (R). *One will have an inequality*

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > V\left(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}\right),$$

on condition that the point (x_i) *be interior to the parallelohedron* $R^{(0)}$.

Let$(\xi_i^{(0)})$ be any point which is interior to the parallelohedron $R^{(0)}$.

Take a point (ξ_i) which is interior to the parallelohedron R and examine a vector $[\xi_i^{(0)}, \xi_i]$ determined by the equations

$$x_i = \xi_i^{(0)} + u(\xi_i - \xi_i^{(0)}) \quad \text{where } 0 \leq u \leq 1\,.$$

A group of the vector $[\xi_i^{(0)}, \xi_i]$ belong to the parallelohedron R^0. Designate,

$$\xi_i' = \xi_i^{(0)} + u_1(\xi_i - \xi_i^{(0)}) \quad \text{where } 0 < u < 1$$

and suppose that the vector $[\xi_i^{(0)}, \xi_i']$ represents the group of the vector $[\xi_i^{(0)}, \xi_i]$ which belong to $R^{(0)}$.

The second group $[\xi_i', \xi_i]$ of the vector $[\xi_i^{(0)}, \xi_i]$ does not possess any point (ξ_i') common to the parallelohedron $R^{(0)}$. The point (ξ_i') belongs to a face $p^{(0)}(\nu)$ of the parallelohedron $R^{(0)}$. One will choose among the parallelohedra which are contiguous by the face $p^{(0)}(\nu)$ a parallelohedron R' which contains a group of the vector $[\xi_i', \xi_i]$.

Designate

$$\xi_i'' = \xi_i + u_2(\xi_i - \xi_i^{(0)}) \quad \text{where } u_1 < u_2 \leq 1$$

and suppose that the vector $[\xi_i', \xi_i'']$ represents a group of the vector $[\xi_i', \xi_i]$ which belongs to the parallelohedron R' and so on.

Let us suppose that one has determined m points of the vector $[\xi_i^{(0)}, \xi_i]$

$$\xi_i^{(k)} = \xi_i^{(0)} + u_k(\xi_i - \xi_i^{(0)})\,, \; (k = 1, 2, \ldots, m) \tag{1}$$

where

$$0 < u_1 < u_2 < \cdots < u_m < 1 \tag{2}$$

which correspond to the vectors $[\xi_i^{(0)}, \xi_i'], [\xi_i', \xi_i''], \ldots, [\xi_i^{(m)}, \xi_i]$ belonging to the parallelohedra

$$R^{(0)}, R', \ldots, R^{(m-1)}, R$$

successively contiguous.

Designate by

$$a_0^{(k)} + \sum a_i^{(k)} x_i = 0, \quad (k = 0, 1, 2, \ldots, m-1)$$

the canonical equation of the face common to the parallelohedra $R^{(k)}$ and $R^{(k+1)}$ which is defined within the parallelohedron $R^{(k)}$.

By virtue of the established definition in Number 32, one will have a formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = V\left(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}\right) + \sum_{k=0}^{m-1}\left(a_0^{(k)} + \sum a_i^{(k)} x_i\right) \tag{3}$$

Examine the sum

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(0)} \quad \text{and} \quad a_0^{(k)} + \sum a_i^{(k)} \xi_i. \quad (k = 0, 1, 2, \ldots, m-1)$$

By virtue of the supposition made, the point $(\xi_i^{(k+1)})$ verifies the equation

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(k+1)} = 0$$

As the point $(\xi^{(k)})$ belongs to the parallelohedron $R^{(k)}$, one will have an inequality

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(k)} \geq 0\,.$$